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Functional Integral Equation for the Complete Effective Action in Quantum Field Theory

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Based on a methodological analysis of the effective action approach, certain conceptual foundations of quantum field theory are reconsidered to establish a quest for an equation for the effective action. Relying on the functional integral formulation of Lagrangian quantum field theory, we propose a functional integral equation for the complete effective action which can be understood as a certain fixed-point condition. This is motivated by a critical attitude toward the distinction, artificial from an experimental point of view, between classical and effective action. While for free field theories nothing new is accomplished, for interacting theories the concept differs from the established paradigm. The analysis of this new concept concentrates on gauge field theories, treating QED as the prototype model. An approximative approach to the functional integral equation for the complete effective action of QED is exploited to obtain certain nonperturbative information about the quadratic kernels of the action. As a particular application the approximate calculation of the QED coupling constant α is explicitly studied. It is understood as one of the characteristics of a fixed point given as a solution of the functional integral equation proposed. Finally, within the present approach the vacuum energy problem is considered, as are possible implications for the concept of induced gravity.

1. INTRODUCTION

Physical reality can be approached by means of quantum field theory from different perspectives. This depends in particular on the kind of information one is interested in extracting in order to solve a problem under consideration. However, the line of attack is also influenced by one's view of the fundamental difficulties in standard quantum field theory (and its generalized concepts such as string theory). To a large extent, these different approaches reflect technical difficulties in fully understanding quantum field-theoretic

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models (in particular, nonperturbatively) rather than real differences in concepts on a fundamental level. However, some of the pioneers of quantum field theory, such as Dirac (1978, 1983) and Feynman (1972, 1985), in particular pointing to the UV divergence problem, always maintained that the right theory had not yet been found. This attitude has apparently not received majority support, although in this respect there does not seem to exist any consensus.[2] We believe this state of affairs justifies a reconsideration of certain conceptual foundations of quantum field theory. This is the purpose of the present paper.

Notwithstanding the above problems, there seems to exist wide agreement that the scattering matrix can be considered as the fundamental object for describing a particular quantum field-theoretic model. The knowledge of the complete scattering matrix is considered equivalent to the solution of a quantum field theory and all interesting information, at least in principle, can be extracted from it. Construction of the scattering matrix can be attempted by different methods. For instance, the so-called S-matrix theory, as studied in the 1950s in reaction to the emergence of the divergence problem in Lagrangian quantum field theory, was designed to find the (finite) scattering matrix from rather general fundamental principles such as causality, unitarity, and Lorentz invariance, using dispersion techniques without making any reference to a Lagrangian underlying the theory (e.g., Brown *et al.*, 1989; Eden *et al.*, 1966). Although quite general and interesting results have been obtained, the principles applied turned out not to be restrictive enough to completely fix the scattering matrix for realistic theories. Nowadays, after the successful reemergence of (renormalizable) Lagrangian quantum field theory at the end of the 1960s, the description of the scattering matrix is supplied in a standard way in terms of the effective action of the theory considered (Slavnov and Faddeev, 1991). In this sense we may view the effective action as the genuine fundamental object of interest and will concentrate on its study in this paper.

Historically, beyond the S-matrix theory, attempts to cure UV divergences by nonlocal field theories have played a significant role since the emergence of the divergence problem in the 1930s [for a review including references see Efimov (1977, 1985); also see Efimov (1987)]. Although it was recognized early that nonlocal field theories may be accompanied by new, perhaps even more unpleasant difficulties, such as with unitarity and (macro-)causality, theoretical thinking in this direction nonetheless continued. Most prominently, string theory, although much more ambitious, can be viewed as a particular way of giving preference to a special kind of nonlocality

[2] For a description of the attitude in one large part of the community see, e.g., Shirkov (1990), and for a more general account with emphasis on the historical development see Brown (1993).

(Eliezer and Woodard, 1989). In recent years, some papers have again dealt with nonlocal quantum gauge field theories (to mention only this subject) where in part the nonlocalities introduced are understood as regulators (Partovi, 1982; Ktorides and Mavromatos, 1985a,b; Karanikas *et al.*, 1986; Karanikas and Ktorides, 1987; Evens *et al.*, 1991; Cornish, 1992; Kleppe and Woodard, 1992). Although having a different aim than dealing with UV divergences, the recently proposed average action concept (Ringwald and Wetterich, 1990; Wetterich, 1991), which is related to the Wilson renormalization group, should also be mentioned here. However, in principle, the drawback of all these nonlocal approaches consists in the arbitrariness in the choice of the nonlocality introduced. So far no unique recipe starting from first principles has been proposed.

The dominant paradigm in the field remains local renormalizable Lagrangian quantum field theory (throughout the paper we will denote it by the term "standard quantum field theory"). However, nonlocality is also a well-known phenomenon in standard quantum field theory because it is a feature of the effective action that can be derived for any quantum field theory (either local or nonlocal) and which also serves (in most cases) as the generating functional of the one-particle-irreducible (1PI) Green functions. In general, the effective action is attributed different meanings by different authors. Some regard the effective action as some low-energy representation of a quantum field theory obtained by integrating out certain (massive) degrees of freedom (e.g., Georgi, 1993), while others consider the effective action as a fully fledged description of the model under investigation from which arbitrary S-matrix elements (related to any observation one might be able to perform) can be derived. We will adopt here the latter view. We take the pragmatic view that the effective action is that object which contains all the information ever to be measured under certain defined circumstances and there is no other (independent) object linking theory to physical reality. The shape of the effective action may of course depend on some of these circumstances (e.g., external conditions). A similar point of view has recently been described with respect to the gravitational effective action by Vilkovisky (1992). The effective action concept we have in mind aims at quantum field-theoretic models, especially those which are realistic like QED, and assumes that certain sectors of physical reality can be described in a consistent way independently of each other. It is therefore quite different from the TOE ('theory of everything') concept often related to superstring theory.

In short, the program of the present article can be described by saying that we intend to find a concept which allows the determination of the structure of the (highly complex) observable 'effective action' without making reference to any other quantity not accessible to observation. In particular, the approach to quantum field theory will be based on a critical attitude

toward the distinction, artificial from an experimental point of view, between the so-called classical action and the effective action. This way we will be led to propose an equation for determining the (finite) effective action, which can be understood as a certain fixed-point condition. It will be an equation for functionals of fields (actions) and is therefore designed to remove the arbitrariness in the choice of the Lagrangian prescribed at the beginning of any field theory (to a certain extent—the field content has to be prescribed as usual). However, this can only be expected to happen for interacting theories, where our approach differs from the established paradigm. For free field theories, where this is not the case, nothing new is accomplished in this respect. As a technical tool we rely on the functional integral formulation of Lagrangian quantum field theory, which seems to be the appropriate and most convenient language for the description of our concept. While nonlocality will be an inherent feature of our approach in most cases, it is by no means the conceptual starting point of the present investigation. Of course, the program as just sketched is an abstract one. However, once we have proposed the general concept it will simply serve us as a guide for finding an appropriate approximative approach to perform explicit calculations (in this article we consider QED as the prototype gauge field theory.).

The outline of the article is as follows. In Section 2 we explain the general concept. First, a quest for an equation for the complete effective action is established and the further discussion then serves to suggest a particular answer by imposing a certain fixed-point condition in terms of a functional integral equation. Section 2 also discusses some features of this equation, among others the relation between standard quantum field theory and the present approach. Section 3 contains the major body of the explicit calculation. The model under investigation is QED in 4D Minkowski (Euclidean) space. Section 3.1 formulates the functional integral equation proposed in Section 2 for gauge field theories. The explicit calculation performed in the remainder of the section relies on an approximative approach. Section 3.2 explains this approach to the functional integral equation for the complete effective action of QED which concentrates on the quadratic kernels of the fermion and gauge field actions. Section 3.3 then presents the explicit calculation. It is further split into several subsections. Section 3.3.1 contains technical details of the functional integration. While for the quadratic kernel of the gauge field action we rely on a certain Ansatz, Section 3.3.2 establishes an integral equation for the quadratic kernel of the fermion action. This integral equation is then approximately solved in Sections 3.3.2.1 and 3.3.2.2 in the asymptotic UV and IR regions, respectively. This analysis yields certain nonperturbative information about the quadratic kernel of the fermion action. Finally, in Section 3.3.3, as a particular application of the present method, the approximate calculation of the QED coupling constant α is explicitly

studied. It is understood as one of the characteristics of a fixed point given as a solution of the functional integral equation proposed. Section 4 closes with a discussion of some aspects of the results obtained and with certain observations concerning the vacuum energy problem in QED as well as related considerations concerning the relevance of the proposed approach to the concept of induced gravity.

2. AN EQUATION FOR THE COMPLETE EFFECTIVE ACTION

As an introductory step, let us begin by displaying key elements of the standard formulation of the effective action (Itzykson and Zuber, 1980). We consider Lagrangian quantum field theory in flat (Minkowski) space-time and in this section we use scalar field theory to pursue the discussion. It is understood that generalization to more complicated theories (in particular, gauge field theories) can be performed merely by standard means. Construction starts with the generating functional of Green functions

$$Z_n[J_n] = C \int D\phi \exp\left\{ i\Gamma_{n-1}[\phi] + i \int dx\, J_n(x)\phi(x) \right\} \tag{2.1}$$

where $\Gamma_{n-1}[\phi]$ is the so-called classical action of the theory and C some fixed normalization constant. (For future purposes we have introduced here an index n. The conventional notation is obtained by setting $n = 1$ and dropping the index 1.) Then, the generating functional of the connected Green functions is

$$W_n[J_n] = -i \ln Z_n[J_n] \tag{2.2}$$

The effective action $\Gamma_n[\bar{\phi}_n]$, which also is the generating functional of the one-particle-irreducible (1PI) Green functions, is obtained as the first Legendre transform of $W_n[J_n]$,

$$\Gamma_n[\bar{\phi}_n] = W_n[J_n] - \int dx\, J_n(x)\bar{\phi}_n(x) \tag{2.3}$$

Here

$$\bar{\phi}_n(x) = \frac{\delta W_n[J_n]}{\delta J_n(x)} \tag{2.4}$$

which in turn leads to

$$\frac{\delta \Gamma_n[\bar{\phi}_n]}{\delta \bar{\phi}_n(x)} = -J_n(x) \tag{2.5}$$

in analogy to the classical field equation for $\Gamma_{n-1}[\phi]$. Equivalently, using the above expressions, we can consider

$$\exp(i\Gamma_n[\bar{\phi}_n]) = C \int D\phi \exp\left\{ i\Gamma_{n-1}[\phi + \bar{\phi}_n] + i \int dx\, J_n(x)\phi(x) \right\} \qquad (2.6)$$

as the defining relation for the effective action, where the r.h.s. of the above equation has to be calculated using a current $J_n(x)$ given by equation (2.5), which is a functional of $\bar{\phi}_n$. Therefore, as the r.h.s. is a functional of both J_n and $\bar{\phi}_n$, equations (2.5) and (2.6) have to be understood as functional integrodifferential equations for determining the (off-shell) effective action (giving an implicit definition only). Equation (2.1) defines a map g_1: $\Gamma_{n-1}[\phi] \to Z_n[J]$ from the class of functionals called classical actions to the class of functionals Z_n. Furthermore, we have mappings g_2: $Z_n[J] \to W_n[J]$ [equation (2.2), single-valued up to the fixing of the sheet of the Riemann surface, uninteresting for the present purpose] and g_3: $W_n[J] \to \Gamma_n[\bar{\phi}]$ [equation (2.3)]. These three maps together define a map $g_3 \circ g_2 \circ g_1 = f$: $\Gamma_{n-1}[\phi] \to \Gamma_n[\bar{\phi}]$ [equation (2.6)] from the set of so-called classical actions to the set of effective actions. In total, this map is unique up to the renormalization problem, which can always be treated in the present context by applying an appropriate regularization procedure for properly handling the divergences. Inasmuch as this map f is constructed explicitly, equation (2.6) is not a genuine equation with possibly a variety of solutions, but rather expresses the image Γ_n of Γ_{n-1} with respect to the map f. Once the functional integral measure is constructed (and typically this is done for a whole class of classical actions and then fixed forever), the classical action Γ_{n-1} uniquely determines the corresponding effective action Γ_n. In other words, the effective action contains no more information than that (implicitly) encoded in the classical action (supplemented by the functional integral measure). The important point to notice here is that the effective action does not appear as an object in its own right, but as a derived quantity only. Basically, we prescribe an effective action in terms of some low-energy information rather than find it from independent (quantum) principles not exhausted by fixing the classical action. Mere reformulations of the calculational tools used to determine the effective action, like Schwinger–Dyson equations, do not change this feature. In the following these observations serve as a starting point for an attempt to find a modified view of quantum field theory based on the effective action concept, whereby justification for the discussion is derived from the still lasting controversy concerning certain conceptual foundations of quantum field theory mentioned in the introduction.

Now, the above methodological insight concerning the standard construction of the effective action may be confronted with the deductive idea often

applied in theoretical physics that the special case (here the classical action) should be derived from the more general one (here the effective action) and not the other way around. In this sense, the complete effective action is the genuine fundamental object to be studied. If, up to further investigation, one is willing to allow that the complete effective action might be an object in its own right,[3] then one has to find a method of determining the complete effective action differing from the established method.[4] There are not many methods available and to use an equation for determining the complete effective action seems to be an approach natural within theoretical physics. Therefore, the above view leads to the task of finding such an equation for the complete effective action. To the best of the author's knowledge, such a question has not been raised in the literature. It should be emphasized that in view of the fundamental role of the effective action in quantum field theory, it deserves to be answered. To give an answer is important independent of the kind it eventually assumes. Even rejection of the question (e.g., by sticking closely to the established formalism) has significant methodological consequences. The search for an equation for the complete effective action needs to be ruled by certain principles. First, solutions of such an equation should be able to reproduce standard quantum field-theoretic results with the required accuracy in order to stay in line with experiment. Obviously, this does not leave much room for an answer different from the known one. Second, the formalism connected with such an equation should differ sufficiently from standard quantum field theory in order to be able to remove known problems, at least in part. Third, any sensible search for an equation for the complete effective action should take into account that the eventual result needs to be sufficiently general in order to be applicable to various situations and has to be restrictive enough at the same time in order to allow concrete information to be derived from it. Basically, there are two different routes to find the particular answer we prefer to the question put forward by proposing a specific equation for the complete effective action. One way is to discuss certain principles to be built in and then to write down an equation which embodies these. The other way, which we will choose, is heuristically to motivate an equation which then will be analyzed with respect to its conceptual content.

Let us consider the map f: $\Gamma_{n-1}[\phi] \to \Gamma_n[\bar{\phi}]$ mapping so-called classical actions to effective actions. Although it is not necessarily well defined for

[3] Of course, any effective action has a certain classical limit, but coincidence of its classical limit with that of another effective action does not necessarily entail identity of both effective actions once this view is accepted.

[4] Certainly, also such a different method which does not start with the classical action may, at the end, lead to the conclusion that classical actions and effective actions are related to each other one-to-one, but then this is a result of the method and not the starting point.

the domain of classical actions (which are local functionals in general), we will not change the map f itself, but instead we will now extend the domain of this map. For this purpose it suffices to mention that the set of so-called classical actions can be considered as a subset of the class of effective actions. From now on we understand the map f as a mapping of the set of effective actions into itself. On the basis of formulas (2.1)–(2.6) we can now define the map f for the extended domain. These equations remain almost unchanged except that we replace the constant C in equations (2.1) and (2.6) by $C \exp(-i\Gamma_{n-1}[0])$. As in equation (2.1), $C = C(\mu)$ is some fixed dimensional normalization constant depending on an arbitrary mass parameter μ and compensating the dimension of the functional integral measure $D\phi$. Changes in μ correspond to changes in the normalization of the vacuum energy connected with $\Gamma_n[0]$. In extending the domain of the map f we introduce an additional normalization factor $\exp(-i\Gamma_{n-1}[0])$. This is not a major point, but one worth appreciating from a conceptual point of view. Classical actions typically are normalized to obey $\Gamma_{n-1}[0] = 0$. Then, equation (2.6) tells us that $\Gamma_n[0]$ completely originates from vacuum fluctuations governed by the classical action Γ_{n-1} (up to some normalization of the vacuum energy fixed for a whole class of actions). By including the additional normalization factor, this principle is generalized to the map f acting in the extended domain and admits calculation of the vacuum energy as usual.[5]

Consider now iterations of the map f leading to some discrete series of effective actions

$$\cdots \overset{f}{\rightarrow} \Gamma_{n-1} \overset{f}{\rightarrow} \Gamma_n \overset{f}{\rightarrow} \Gamma_{n+1} \overset{f}{\rightarrow} \cdots$$

Obviously, the most interesting question one may ask with respect to the iterations of the map f is whether it has any fixed point. It should be expected that the fixed-point condition for the map f is not trivially fulfilled for any arbitrary action and should distinguish certain (complete) effective actions. Now, we propose that the fixed-point condition for the map f defined above yields the equation for the complete effective action we are seeking. The equation for the complete effective action which is equivalent to the fixed-point condition for the map f reads

$$\exp(i\Gamma[\bar{\phi}]) = C \exp(-i\Gamma[0]) \times \int D\phi \exp\left\{ i\Gamma[\phi + \bar{\phi}] + i \int dx\, J(x)\phi(x) \right\} \qquad (2.7)$$

[5] Having in mind standard quantum field theory, of course, here we refer to vacuum energy modifications under external conditions.

where

$$J(x) = -\delta\Gamma[\bar{\phi}]/\delta\bar{\phi}(x) \tag{2.8}$$

Equations (2.7) and (2.8) together define a genuine functional integrodifferential equation for determining the complete (off-shell) effective action Γ of a quantum field theory. Of course, this equation needs to be supplemented by additional information to specify the particular conditions under which it should be solved. Accumulated experience in quantum field theory tells us that, in general, solutions of equation (2.7)—if there are any at all—should be expected to be nonlocal and nonpolynomial functionals Γ of the field ϕ. Optimistically, one might think that the above equation for the complete effective action is sufficiently restrictive in the case of interacting theories to enable us not only to find the structure of the effective action, but also to determine the dimensionless parameters it contains (e.g., coupling constants and mass ratios). Of course, the eventual range of theories to which it can be applied remains to be explored. However, it seems that at least any theory which cannot be understood as being induced by some more fundamental one should be subject to the concept of the complete effective action discussed above. Before we analyze equation (2.7) from the conceptual side, let us ask whether it has any solution at all. The answer is that any free field theory solves equation (2.7) (in saying so, of course, we neglect the vacuum energy problem). For free field theories the new approach proposed completely agrees with the standard formulation of quantum field theory. However, the former obviously differs from the latter for interacting theories. It remains to be seen whether there exists any interacting field theory which solves equation (2.7).

The S-matrix which finally describes the results of physical measurements is just the S-matrix of that nonlocal and nonpolynomial field theory which turns out to be a solution of equation (2.7). Such requirements for the S-matrix, such as unitarity and causality, have to be studied for that particular field theory. However, those properties have been shown to hold for fairly large classes of nonlocal and nonpolynomial field theories (Efimov, 1977, 1985; also see Alebastrov and Efimov, 1973, 1974; Efimov, 1975), providing ground for certain optimism in this respect. However, it remains to be seen in future explicit investigations whether form factors of the solutions of equation (2.7) belong to those function classes (certain entire analytic functions) for which unitarity and causality of the S-matrix has already been demonstrated. In Minkowski space solutions of equation (2.7) also should be expected to exhibit certain imaginary parts, which, however, do not in principle pose any problem with respect to unitarity and causality (Veltman, 1963). Finally, beyond the functional integral approach on which our discussion is based, nonlocal and nonpolynomial field theories can equivalently

also be dealt with within operator quantization (Efimov, 1977, 1985), but it is technically much more involved and we feel that it is not well suited for our purposes.

Now we study equation (2.7) with respect to its methodological content. The proposed equation for the complete effective action is exclusively expressed in terms of an observable (at least, in principle) quantity, namely the complete effective action, which should be finite, of course. This specifies the concept of renormalizable quantum field theory by relying on observable objects only (bare and dressed/renormalized quantities agree here). In this context one may wonder whether the conceptual distinction between classical action and effective action is really a productive one. Although any theoretician may extract the classical limit from any solution of equation (2.7), one may justifiably ask what this tells an experimental physicist. In reality, vacuum fluctuations cannot be switched off (at best, they can be modified) and the experimentally relevant quantity is the effective action. Rather, the experimental physicist is interested in the leading (low-energy, long-distance, low-intensity) terms of the derivative expansion of the effective action, but these do not necessarily coincide with what is called the classical action, although they will contain it in most cases. In view of our equation for the complete effective action, it is also of limited sense to ask which effective action term is induced and which is not, because equation (2.7) is a self-consistency condition.

Continuing the above consideration, it should be mentioned that already in standard quantum field theory there is no difference in principle between a certain mode of vacuum fluctuations and macroscopic (external) fields. This is reflected in the insight that the effective action has a dual nature, namely, on one hand it is considered as the action governing the behavior of macroscopic (external) fields, and on the other hand it is the generating functional of 1PI Green functions playing a central role in describing vacuum fluctuations. In addition, any particular mode of vacuum fluctuations is acting in the background of all of them and merely experiences their total effective impact as described by the complete effective action. Therefore, the path integral construction should not rely on the classical action governing the weight of each path (mode) as is done in standard quantum field theory, but the weight of each path (mode) should be determined by the complete effective action expressing the vacuum properties in total. Of course, this involves a certain self-referentiality which finds its adequate formulation in terms of a genuine equation. In conclusion we may say that equation (2.7) is the theoretical expression of the dual nature of the complete effective action being effective action and generating functional at the same time. In other words, vacuum fluctuations are governed by one and the same action, like macroscopic phenomena.

Having obtained a certain insight into the principles embodied in the proposed equation for the complete effective action, let us turn in the following to methods for its solution. To expect any final answer would not be realistic; instead we discuss some aspects which come to mind immediately. Although there is no quick answer, one may ask whether the map f has something like a contraction property in a certain neighborhood of a solution of equation (2.7). If this is the case, one could attempt its solution by iteration. With this in mind we will see how standard quantum field theory relates to the present formulation. The standard formulation of quantum field theory can be viewed as a first iteration of the map f starting from a certain low-energy (local) approximation (the so-called classical action) to the complete effective action. This can be considered as a natural starting point which is expected to be close to a fixed point of the map f for 'experimental' reasons. However, it is clear that in view of equation (2.7), even the 'complete' (assuming we had summed the usual perturbation theory) effective action of standard quantum field theory given by equation (2.6) is not the complete one in the sense of equation (2.7), but remains an approximation. The approximation method represented by standard local quantum field theory works reasonably well in lower spacetime dimensions, with considerable effort in four dimensions, but it becomes badly defined for most theories in higher dimensions. So, one may consider the properties of a theory with respect to renormalization as information about the possible quality of an approximate solution of equation (2.7) obtained from some local Ansatz by iteration of the map f. Quantization of a classical theory can be understood as a method for approximately solving equation (2.7). However, simple extrapolation of the classical Lagrangian to arbitrarily high energies leads to the well-known UV divergences.

For practical (i.e., calculational) purposes the map f is not a very convenient one. Instead one may use a somewhat simpler map $\tilde{f}$ which differs from f, but, as one may see easily from equation (2.7), has the same set of fixed points as f. This simpler map $\tilde{f}$: $\Gamma_{n-1} \to \Gamma_n$ is given by

$$\exp(i\Gamma_n[\bar{\phi}]) = C \exp(-i\Gamma_{n-1}[0]) \times \int D\phi \exp\left\{ i\Gamma_{n-1}[\phi + \bar{\phi}] + i \int dx\, J_{n-1}(x)\phi(x) \right\} \quad (2.9)$$

The advantage of this formula is that it provides us with a compact and explicit representation of the $\tilde{f}$-image of Γ_{n-1}. However, in general an image of this map $\tilde{f}$ will not have the property of being a generating functional of 1PI Green functions.

In concluding this section, let us express our view that the proposed equation for the complete effective action embodies features which seem reasonable and interesting from a physical point of view and also offers a

guideline for a reevaluation of the established technical approach to quantum field theory and eventually its appropriate modification. From now on we simply will take equation (2.7) as granted and consider it as the starting point for further analysis.

3. QED. AN APPROXIMATIVE APPROACH TO THE EQUATION FOR THE COMPLETE EFFECTIVE ACTION

Among the numerous possible directions to explore the concept proposed in the preceding section, the investigation of QED in 4D seems to be the most interesting and important one. On one hand, QED is the theory where new theoretical ideas can be confronted with experiment (i.e., physical reality) most easily, and on the other hand, as the prototype gauge field theory, it tests the ability to deal with certain technical difficulties not met in scalar model field theories, for example. To study QED we will therefore begin this section with a formulation of the functional integral equation for the complete effective action in gauge field theories supplemented by a discussion of certain special features relevant to this class of theories. Besides structural investigation of the equation for the complete effective action, the question of most interest appears to be whether the proposed approach will enable us to extract concrete information which is not at all, or not easily, obtainable by established standard methods. The aim of this section is to demonstrate that the present approach indeed allows us explicitly to find certain information about the complete effective action of QED, which, in addition, can be seen to be of a nonperturbative nature. Of course, the concrete study of the equation for the complete effective action of QED cannot be expected to be rigorous for the time being. It will be necessary to apply an approximation, which, however, in certain respects should circumvent some of the problems appearing in standard quantum field theory. In particular, as far as is possible, we will take care that no inappropriate approximation giving rise to UV divergences is introduced. Although most of the approximations we will exploit in the explicit calculations of this section can be expected to be reasonable for small values of the QED coupling constant α, the explicit calculation we will undertake has to be understood in the first place as a model game to test in principle the calculational accessibility of the concept proposed. As a particular application of the new concept, we will explicitly study how to determine the coupling constant α (i.e., the theoretical value of the fine structure constant), understood as one of the characteristics of a fixed point of the map f. This is done using certain simple approximations (capable of future improvement), which at the end, however, will turn out somewhat too simple to succeed numerically.

3.1. The Functional Integral Equation for Gauge Field Theories

In this subsection we study the equation for the complete effective action proposed in Section 2 in the case of gauge field theories. Although we will have in mind gauge field theories in general, here we restrict ourselves to QED and only comment on the case of non-Abelian gauge theories. In doing so it is understood that the Faddeev–Popov procedure used in standard quantum field theory for defining the functional integral measure can be applied in a slightly generalized way also in the present context, in particular, taking into account that, in general, solutions of equation (2.7) are nonlocal and the gauge condition to be chosen will be, for convenience, likewise nonlocal.

We start by defining the generalized map f for QED. The generating functional Z of the Green functions is

$$\begin{aligned} Z_n[J_n, \bar{\eta}_n, \eta_n] &= C \exp(-i\Gamma_{n-1}[0, 0, 0]) \\ &\times \int D[a_\mu]\, D\psi\, D\bar{\psi} \exp(i\Gamma_{n-1}[a, \psi, \bar{\psi}]) \\ &\times \exp\Big\{ i\Gamma_{gf}[a] \\ &+ i \int d^4x\, [J_{n\mu}(x)a^\mu(x) + \bar{\eta}_n(x)\psi(x) + \bar{\psi}(x)\eta_n(x)] \Big\} \end{aligned} \tag{3.1}$$

where

$$\Gamma_{gf}[a] = -\frac{1}{2\lambda} \int d^4y\, (F[a; y])^2 \tag{3.2}$$

$$F[a; y] = \int d^4x\, n_\mu(y - x)a^\mu(x) \tag{3.3}$$

As usual, Γ_{gf} is a gauge-breaking term containing a linear, homogeneous functional F of a_μ (for the moment n_μ is any arbitrary but appropriately chosen vector-valued distribution) and the brackets in $D[a_\mu]$ in equation (3.1) indicate that the Faddeev–Popov determinant has to be taken into account.[6]

[6] It is an almost trivial factor for Minkowski space QED, but already at finite temperature it becomes important. In addition, always having in mind possible generalization to non-Abelian gauge theories, it serves as a reminder for this complication then to be considered.

Here Γ_{n-1} belongs to the class of gauge-invariant effective actions. Then, the W-functional is given by

$$W_n[J_n, \bar{\eta}_n, \eta_n] = -i \ln Z_n[J_n, \bar{\eta}_n, \eta_n] \tag{3.4}$$

and the image of Γ_{n-1} is

$$\begin{aligned}\Gamma_n[A_n, \Psi_n, \bar{\Psi}_n] &= W_n[J_n, \bar{\eta}_n, \eta_n] \\ &\quad - \int d^4x \, [J_{n\mu}(x)A_n^\mu(x) + \bar{\eta}_n(x)\Psi_n(x) + \bar{\Psi}_n(x)\eta_n(x)]\end{aligned} \tag{3.5}$$

Again, we have the relations

$$A_{n\mu}(x) = \frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta J_n^\mu(x)}, \qquad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta A_n^\mu(x)} = -J_{n\mu}(x) \tag{3.6}$$

$$\Psi_n(x) = \frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta \bar{\eta}_n(x)}, \qquad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \Psi_n(x)} = \bar{\eta}_n(x) \tag{3.7}$$

$$\bar{\Psi}_n(x) = -\frac{\delta W_n[J_n, \bar{\eta}_n, \eta_n]}{\delta \eta_n(x)}, \qquad \frac{\delta \Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]}{\delta \bar{\Psi}_n(x)} = -\eta_n(x) \tag{3.8}$$

Now, performing shifts in the integration variables, we find

$$\begin{aligned}&\exp(i\Gamma_n[A_n, \Psi_n, \bar{\Psi}_n]) \\ &\quad = C \exp(-i\Gamma_{n-1}[0, 0, 0]) \\ &\qquad \times \int D[a_\mu] \, D\psi \, D\bar{\psi} \exp(i\Gamma_{n-1}[a + A_n, \psi + \Psi_n, \bar{\psi} + \bar{\Psi}_n]) \\ &\qquad \times \exp\Big\{ i\Gamma_{gf}[a + A_n] \\ &\qquad + i \int d^4x \, [J_{n\mu}(x)a^\mu(x) + \bar{\eta}_n(x)\psi(x) + \bar{\psi}(x)\eta_n(x)] \Big\}\end{aligned} \tag{3.9}$$

describing the map f from the gauge-invariant effective action Γ_{n-1} to its image Γ_n. From the discussion leading to the background field method in gauge field theories we know that Γ_n is in general not gauge invariant because, as one easily recognizes from equation (3.9), the shift in the gauge field integration interferes with the gauge-fixing term for the quantum fluctuations. This is remedied in standard quantum field theory by starting in equation (3.1) with a modified gauge-fixing term $\Gamma_{gf}[a - A]$ and fixing the field A_μ

to obey $A_\mu = A_{n\mu}$ (cf. Abbott, 1981, and references therein). However, in our approach the application of this procedure would entail that the map f (in particular, the gauge condition for the quantum fluctuations) had to be modified in each iteration step depending on the actual shape (gauge) of $A_{n\mu}$, i.e., of $F[A_n - A; y]$. While in standard quantum field theory $A_{n\mu}$ can be understood as some fixed background field (essentially, this makes the background field method acceptable), our situation is worse in this respect because $A_{n\mu}$ also contains pieces of arbitrary vacuum fluctuations to be integrated over later. There is only one safe way to ensure that the gauge for $A_{n\mu}$ and that for the vacuum fluctuations a_μ do not interfere in a gauge-dependent way (i.e., that the shift in the argument of the gauge field integration does not interfere with the gauge-fixing term), namely one has to choose for $A_{n\mu}$ the gauge

$$F[A_n; y] = 0 \tag{3.10}$$

If $A_{n\mu}$ is a sum of independent pieces, condition (3.10) applies to each component because F is linear and homogeneous. Now, as already mentioned, in general, $A_{n\mu}$ contains pieces of vacuum fluctuations to be integrated over in further iterations; consequently, we have to impose condition (3.10) also on these vacuum fluctuations. This argument of course applies to each iteration step of the map f and therefore the only consistent gauge is the generalized Landau gauge $\lambda = 0$. So a 'sharp' gauge has to be imposed on all gauge fields, on external fields as well as on vacuum fluctuations, i.e., the whole system of functional relations is bound to one definite gauge. Of course, the gauge functional F can be chosen as convenience may require and the full gauge-invariant effective action Γ_n consequently is obtained by letting F vary.

At this point it seems appropriate to mention that in the past decade the effective action concept has received interest from the point of view of its invariant geometrical formulation. This is an important step in ensuring the physical relevance of the effective action because its physical consequences should not depend on the particular choice of coordinates for the field variables. Initial work in this direction traces back to Vilkovisky (1984a,b) and DeWitt (1987). For a recent discussion of the geometrical effective action see Camblong and Ordóñez (1992) and for a review including further references see Buchbinder *et al.* (1992). For the purpose of the present article (to reduce the complexity of the considerations) we simply have bypassed the subject so far (implicitly) maintaining that always those field coordinates are applied in terms of which the formalism takes its naive (nongeometrical) shape. For gauge field theories, a main concern of the unique (geometrical) effective action concept, we have seen above that the generalized Landau gauge is the only sensible gauge. Inasmuch as for gauge field theories the geometrical effective action has been found to agree with the naive one

(calculated by means of the standard background field method) exactly for generalized Landau gauge, we can feel free to ignore the subject also in this context (Fradkin and Tseytlin, 1984; Rebhan, 1987; Nachbagauer *et al.*, 1989).

From equation (3.9) we now read off the equation for the complete (gauge-invariant) effective action of QED,

$$\begin{aligned}
&\exp(i\Gamma[A, \Psi, \bar{\Psi}]) \\
&\quad = C \exp(-i\Gamma[0, 0, 0]) \\
&\qquad \times \int D[a_\mu]\, D\psi\, D\bar{\psi} \exp(i\Gamma[a + A, \psi + \Psi, \bar{\psi} + \bar{\Psi}]) \\
&\qquad \times \exp\Big\{ i\Gamma_{gf}[a] \\
&\qquad + i \int d^4x\, [J_\mu(x)a^\mu(x) + \bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x)] \Big\} \qquad (3.11) \\
&\qquad\qquad F[A; y] = 0, \qquad \lambda \to 0
\end{aligned}$$

In any explicit calculation we will always leave the gauge parameter λ unfixed because this allows us to keep better track of the terms involved, and in the final results one may simply set $\lambda = 0$ to find the correct answer.

Finally, let us mention that Ward–Takahashi identities can be studied within the present formulation of QED, which, however, retain their standard shape (Scharnhorst, 1993, Section 3.1). Furthermore, one may convince oneself that also in non-Abelian gauge field theories the derivation of generalized Ward identities [i.e., Slavnov–Taylor identities (Slavnov and Faddeev, 1991, Section IV.7] remains unchanged and they also hold at each step of any iteration of the map f. Violation of these (generalized) Ward identities (e.g., if anomalies occur) means that the equation for the complete effective action of such a theory will not have any solution. To see this, note that the existence of an anomaly would entail that the image $\Gamma_n = f(\Gamma_{n-1})$ of an action has a different behavior than its counterimage Γ_{n-1}, blocking any attempt to solve the equation. In this sense, the well-known model-building requirement of anomaly cancellation (e.g., Ryder, 1985, Section 9.10) can be understood as a solvability condition for the functional integral equation for the complete effective action of a theory under consideration.[7]

3.2. The Approximative Approach

After having established in the preceding section the equation for the complete effective action in QED, we now investigate it by means of some

[7] Of course, as in standard quantum field theory, this concerns only dynamical fields.

rough approximation, which, however, is a straightforward generalization of standard QED. The approximative approach is as follows. We study one iteration of the map f starting from a certain Ansatz Γ_{I} which is mapped by means of f to its image Γ_{II}. The gauge-invariant Ansatz for Γ_{I} is chosen as a natural generalization of the so-called classical action Γ_0 (to obtain this replace d_{I}, a_{I}, b_{I} by delta functions in the equations below), which is the starting point for standard QED perturbation theory,

$$\Gamma_{\mathrm{I}}[A, \Psi, \bar{\Psi}] = \Gamma_{\mathrm{I}}^{G}[A] + \Gamma_{\mathrm{I}}^{f}[A, \Psi, \bar{\Psi}] \tag{3.12}$$

$$\Gamma_{\mathrm{I}}^{G}[A] = \frac{1}{2}\int d^4x\, d^4x'\, A^{\mu}(x) \times [g_{\mu\nu}\, {}_x\Box - {}_x\partial_{\mu}\, {}_x\partial_{\nu}]\, d_{\mathrm{I}}(x - x')\, A^{\nu}(x') \tag{3.13}$$

$$\Gamma_{\mathrm{I}}^{f}[A, \Psi, \bar{\Psi}] = \int d^4x\, d^4x'\, \bar{\Psi}(x) \exp\left[ie \int_x^{x'} dy_{\mu}\, A^{\mu}(y)\right] \times [a_{\mathrm{I}}(x - x')(i\, \partial\!\!\!/_{x'} - e A\!\!\!/(x')) - m b_{\mathrm{I}}(x - x')]\Psi(x') \tag{3.14}$$

m is the electron mass, d_{I}, a_{I}, and b_{I} are functions (distributions) that are arbitrary for the moment, and the gauge functional F appearing in equation (3.9) is to be chosen later in a way appropriate and convenient for the explicit calculation.[8] Furthermore, the line integration in the phase factor in equation (3.14) is understood to be performed along a straight line connecting starting and endpoints. Equation (3.14) is written in such a form as to keep contact with standard QED ($\tilde{a}_{\mathrm{I}} = \tilde{b}_{\mathrm{I}} \equiv 1$) as closely as possible. Finally, the equation for the complete effective action (3.11) will be taken into account in such a way that we require at the end $d_{\mathrm{I}} = d_{\mathrm{II}}$, $a_{\mathrm{I}} = a_{\mathrm{II}}$, $b_{\mathrm{I}} = b_{\mathrm{II}}$, at least in some approximation. All new structures of Γ_{II} not appearing in the Ansatz Γ_{I} will be viewed as induced ones within this approximation and remain beyond the scope of present interest.

It should be mentioned that an Ansatz similar to equation (3.14) (with $a_{\mathrm{I}} = b_{\mathrm{I}}$) was unsuccessfully explored earlier within the framework of nonlocal QED by Chrétien and Peierls (1954) (see also Peierls, 1954, 1991). For a discussion and an explanation of the failure of the attempt see Scharnhorst

[8] In general, we will alternatively write $l(x)$ or $l(r)$, $r = -m^2x^2$, for one and the same function, which, however, will not lead to any confusion in the context used. [All functions $l(x)$ we study depend on x via x^2 only. l stands here for d, a, b.] Fourier transforms are defined for $l(x)$ by

$$l(x) = \int \frac{d^4p}{(2\pi)^4}\, e^{ipx} \tilde{l}(p)$$

and equivalently we use the notation $\tilde{l}(p)$ and $\tilde{l}(s)$, $s = -p^2/m^2$, for one and the same function.

(1995). With reference to Chrétien and Peierls (1954), the action (3.14) has also recently been studied in a different context (effective Lagrangians in nuclear theory) than ours (Ohta, 1990; Bos *et al.*, 1991; Terning, 1991).

Having spelled out our general kind of approximative approach, we need now to translate it into operational terms which are fundamental to the explicit calculation we are aiming at. So far, d_I, a_I, and b_I are understood as completely arbitrary and clearly it is difficult to perform an explicit calculation based on such a general Ansatz. Therefore, below first we will discuss whether the most general Ansatz for d_I, a_I, and b_I can sensibly be restricted to a certain subclass in which the final solution can be sought. Of particular interest is whether these distributions can adequately be modeled by means of local operators. Let us start with the consideration of a_I, b_I characterizing the fermion action Γ_I^F.

One of the crucial solvability conditions of equation (3.11) is that the map f should not violate gauge invariance. This in particular entails that the map f must not induce any mass term for the gauge field A_μ. Even a finite nonvanishing coefficient of such a mass term is not allowed, not to mention infinite ones, which are pushed aside in standard QED by applying a gauge-invariant regularization. Inasmuch as here we are aiming at finite solutions of the equation for the complete effective action (i.e., some approximation to it) even in a gauge-noninvariant regularization scheme (like cutoff regularization), mass terms should not survive after lifting the regularization. An analysis within cutoff regularization of the mass term induced by the fermionic part of the functional integral yields the following conditions from the requirement that the induced mass term should vanish when removing the cutoff [cf. Scharnhorst (1995), equations (3.7), (3.8); $s = -p^2/m^2$]:

$$\tilde{a}_I(s) \overset{s\to\infty}{\sim} O(s^\kappa) \tag{3.15}$$

$$\tilde{b}_I(s) \overset{s\to\infty}{\sim} \text{const} + O(s^\kappa), \qquad \text{const} \neq 0, \quad \kappa < -1 \tag{3.16}$$

From these relations one recognizes that $\tilde{a}_I$ and $\tilde{b}_I$ should behave differently for $s \to \infty$, i.e., they cannot be identical. This requirement is in line with results for the fermion self-energy calculated in lowest order of standard QED perturbation theory, where $\tilde{a}$ and $\tilde{b}$ already differ (e.g., Itzykson and Zuber, 1980). Although (3.15), (3.16) provide us with certain expectations for the UV behavior of a_I and b_I, this result does not seem to improve our situation. Even worse, it indicates that a_I and b_I cannot adequately be approximated by any local operator Ansatz because it would exhibit an unacceptable UV behavior. So we conclude that for the moment a_I and b_I should indeed be kept arbitrary and the hope for simplifying our Ansatz is

exclusively placed on the kernel of the gauge field action Γ_Γ^G, which we will discuss now.

The first requirement [Section 3.3.1, equation (3.33)] is that we expect the (time-integrated) self-energy

$$\frac{1}{2}\int d^4y\, d^4y'\, \bar{J}_\mu(x, x'; y) D_\Gamma^{\mu\nu}(y - y') \bar{J}_\nu(x, x'; y') \tag{3.17}$$

($D_\Gamma^{\mu\nu}$ is the photon propagator derived from the action $\Gamma_\Gamma^G + \Gamma_{gf}$) of a charged point particle represented by the current

$$\bar{J}_\mu(x, x'; y) = e\int_0^1 d\tau\, \dot{z}_\mu \delta^{(4)}(z(\tau) - y) \tag{3.18}$$

$$z_\mu(\tau) = (x' - x)_\mu \tau + x_\mu$$

and propagating over a finite time interval to be finite. This is needed in order to properly define the map f. The above requirement yields the condition

$$\tilde{d}_\Gamma(s) \overset{s\to\infty}{\sim} O(s^\kappa), \qquad \kappa > 1/2 \tag{3.19}$$

Taking condition (3.19) into account is sufficient for most of the explicit calculation we are attempting. However, it turns out that in finally imposing our approximation to the fixed-point condition for the map f and then searching for a solution, we need to consider further requirements in order to find one.[9] Specifically, a solution correct in the asymptotic UV region can only be found if the photon propagator $D_\Gamma^{\mu\nu}(x)$ is finite in the coincidence limit $x \to 0$. This entails for the kernel of the gauge field action the stronger requirement

$$\tilde{d}_\Gamma(s) \overset{s\to\infty}{\sim} O(s^\kappa), \qquad \kappa > 1 \tag{3.20}$$

We see that d_Γ characterizing the kernel of the gauge field action should behave qualitatively quite differently than do a_Γ and b_Γ defining the kernel of the fermion action. Conditions (3.19), (3.20) induce justified hope that d_Γ can indeed be modeled by a local operator. Inasmuch as to respect condition (3.19) is sufficient for most of the further explicit calculation (i.e., in particular for the analysis of the asymptotic IR region), we choose the Ansatz

$$d_\Gamma(x) = \left[1 + \beta \frac{\Box}{m^2}\right]\delta^{(4)}(x) \tag{3.21}$$

where β is an arbitrary real (positive) constant parametrizing the Ansatz.[10]

[9] More precisely, this concerns the integral equation for the kernel of the fermion action to be studied further below (Section 3.3.2.1).

[10] We have immediately normalized the first term to 1, freezing the arbitrariness against (finite) gauge field renormalizations that the formalism admits.

The analysis of the asymptotic IR region will be merely independent of further terms to be introduced in (3.21) to satisfy (3.20) and therefore they are ignored in the present Ansatz for calculation simplicity. Of course, the Ansatz introduces an additional (spurious) pole at $p^2 = \beta^{-1}m^2$ in the momentum-space photon propagator representation. However, we will not be worried by this fact, because we simply see equation (3.21) as a model representation of an unknown and possibly complicated kernel of the gauge field action, and so it cannot be expected to be free of perhaps unpleasant properties in any respect. Also, equation (3.21) can be understood as some low-energy (i.e., IR) approximation that, however, can safely be extended to arbitrarily high energies without severely misrepresenting the required true UV behavior. For a discussion of some features and drawbacks of the particular model Ansatz (3.21) see Pais and Uhlenbeck (1950), Barcelos-Neto *et al.* (1991), and references therein. The analysis of the asymptotic UV region will not demand any further explicit knowledge of the photon propagator beyond condition (3.20), so that the Ansatz (3.21) can be used for most of the further calculation (which focuses on the IR analysis) and does not need to be supplemented by any specific UV Ansatz.

We are prepared now to define in some detail the approximation strategy to be followed in the explicit calculation. In order to reduce the calculational complexity, we will use the map $\tilde{f}$ (i.e., source terms are given by $\Gamma_{\rm I}$ and not by $\Gamma_{\rm II}$) instead of the map f. In practice, $\tilde{f}$ will be slightly modified still further, as we will explain in Section 3.3.1. The local operator Ansatz (3.21) for the kernel of the gauge field action admits the following procedure for applying the map $\tilde{f}$. First, starting from $\Gamma_{\rm I}$ with equation (3.21) inserted, we will perform the functional integration over the gauge potentials. This can be done exactly, independent of the Ansatz (3.21). Then, we perform the integration over the fermion fields, and consequently we impose the fixed-point approximation $a_{\rm I} = a_{\rm II}$, $b_{\rm I} = b_{\rm II}$. These integral equations have to be solved. In practice, solution of these coupled integral equations can be attempted in a certain approximation only. Specifically, we will explicitly solve them in the asymptotic UV region and in the asymptotic IR region, respectively. Solutions a, b of these integral equations are still parametrized by α ($\alpha = e^2/4\pi$),[11] while we find that the parameter β (of the kernel of the gauge field action) has to be considered as a function of α in order to find any consistent solution at all. However, we have yet to impose the third condition $d_{\rm I} = d_{\rm II}$.

[11] Let us assume that there is an unique solution a, b only, which is supported by the explicit calculation to be discussed below.

The fermionic integration finally has induced a contribution $\Delta\Gamma_{\mathrm{I}}^{G}$ to the gauge field action as follows[12]:

$$\Delta\Gamma_{\mathrm{I}}^{G}[A] = \frac{\alpha}{4\pi}\int d^4x\, A^{\mu}(x)[g_{\mu\nu}\Box - \partial_{\mu}\partial_{\nu}]\left[C_{1a} + C_{2a}\frac{\Box}{m^2} + \cdots\right]A^{\nu}(x) \tag{3.22}$$

C_{1a}, C_{2a} are functionals of the distributions a and b. Therefore, they can also be viewed as certain functions of α and of the parameter $\beta(\alpha)$. For the moment let us vary the parameter β independently of α, although we believe that the necessity to consider the parameter β as a function of α in the course of solving the integral equation for the quadratic kernel of the fermion action is not bound to the particular method we will apply. The condition $d_{\mathrm{I}} = d_{\mathrm{II}}$ then reads

$$C_{1a}(\alpha, \beta) = 0 \tag{3.23}$$

$$C_{2a}(\alpha, \beta) = 0 \tag{3.24}$$

and both these equations define an implicit function $\alpha(\beta)$ [or $\beta(\alpha)$], i.e., certain curves in the α, β plane. The crossing points of these curves correspond to the set of allowed values (α, β). The functional C_{1a} has been explicitly calculated (Scharnhorst, 1995) with considerable effort in one-loop approximation only (i.e., taking into account the quadratic kernel of the fermion action in the presence of an arbitrary gauge potential). To determine C_{2a} in one-loop approximation along the same lines is a trivial but extremely laborious task reserved for the future. However, if, as mentioned, the parameter β has to be viewed as a function of α in advance of imposing $d_{\mathrm{I}} = d_{\mathrm{II}}$, equations (3.23) and (3.24) cannot be satisfied simultaneously anyway (to expect that they are degenerate seems not to be very realistic). Requiring that at least in the asymptotic IR (long-distance, long-wavelength) region the fixed-point condition should be fulfilled, we choose equation (3.23) as the condition to be respected. So, in principle the equation

$$C_{1a}(\alpha, \beta(\alpha)) = 0 \tag{3.25}$$

allows us to determine the QED coupling constant α within the present approximative approach. It is clear that the above method can easily be accommodated to the inclusion of additional terms in the Ansatz (3.21).

We are now equipped with a plan for the explicit calculations, and will proceed along the lines just discussed.

[12]Gauge-noninvariant structures do not occur because the solutions a and b exhibit a UV behavior, as will be shown, preventing those from occurring even in a gauge-noninvariant regularization (at removing the cutoff).

3.3. Explicit Calculation

3.3.1. Performing the Functional Integration

According to our approximation strategy, first we have to calculate the functional integral [cf. equations (2.9), (3.9)]

$$\exp(i\Gamma_{II}[A, \Psi, \bar{\Psi}])$$

$$= C \int D[a_\mu]\, D\psi\, D\bar{\psi} \exp(i\Gamma_I[a + A, \psi + \Psi, \bar{\psi} + \bar{\Psi}])$$

$$\times \exp\Big\{ i\Gamma_{gf}[a]$$

$$+ i \int d^4x\, [J_{I\mu}(x)a^\mu(x) + \bar{\eta}_I(x)\psi(x) + \bar{\psi}(x)\eta_I(x)] \Big\} \tag{3.26}$$

with

$$\frac{\delta\Gamma_I[A, \Psi, \bar{\Psi}]}{\delta A^\mu(x)} = -J_{I\mu}(x) \tag{3.27}$$

$$\frac{\delta\Gamma_I[A, \Psi, \bar{\Psi}]}{\delta\Psi(x)} = \bar{\eta}_I(x) \tag{3.28}$$

$$\frac{\delta\Gamma_I[A, \Psi, \bar{\Psi}]}{\delta\bar{\Psi}(x)} = -\eta_I(x) \tag{3.29}$$

inserted. In calculating $J_{I\mu}$, we may neglect the term stemming from Γ_I^F because in Γ_{II} it gives rise to fermion interactions only.[13] Furthermore, by using a partial integration, we rewrite equation (3.14) in the following manner [for the definition of $\bar{J}$ see equation (3.18)]:

$$\Gamma_I^F[A, \Psi, \bar{\Psi}] = \int d^4x\, d^4x'\, \bar{\Psi}(x) \exp\Big[i \int d^4y\, \bar{J}_\mu(x, x'; y)A^\mu(y) \Big]$$

$$\times [i\, \partial\!\!\!/_x a_I(x - x') - m b_I(x - x')]\Psi(x') \tag{3.30}$$

This will allow us to represent the result of the gauge field integration, which is to be performed first, in a very convenient way. To perform the gauge

[13] Incidentally, it should be noted that reasoning leading to this fact also makes use of Furry's theorem (i.e., an appropriate generalization of it) which applies to our situation. It excludes a closed fermion loop tadpole contribution.

field integration, we temporarily expand the term $\exp(i\Gamma_1^F)$ in equation (3.26) in a power series,

$$\exp(i\Gamma_1^F) = 1 + i\Gamma_1^F - \tfrac{1}{2}(\Gamma_1^F)^2 + \cdots \tag{3.31}$$

which is a very natural procedure in view of the Grassmann integration. It turns out that the result of the gauge field integration can be given as an infinite sum of Gaussian integrals. Each term of this sum corresponds to a certain power n of Γ_1^F and contains the expression

$$\int D[a_\mu] \exp\left\{ i\Gamma_1^G[a] + i\Gamma_{gf}[a] + i \sum_{k=1}^{n} \int d^4y\, \bar{J}_\mu(x_k, x_k'; y) a^\mu(y) \right\} \tag{3.32}$$

where the arguments $\{x_k, x_k'\}$ refer to the integration variables in the kth copy of Γ_1^F. Performing the Gaussian integration, we find that (3.32) reads

$$C \exp\left\{ -\frac{i}{2} \sum_{k=1}^{n} \sum_{l=1}^{n} \int d^4y\, d^4y'\, \bar{J}_\mu(x_k, x_k'; y) D_1^{\mu\nu}(y - y') \bar{J}_\nu(x_l, x_l'; y') \right\} \tag{3.33}$$

Terms with $k = l$ are self-energy contributions, while off-diagonal terms of the double sum in the exponent generate fermion interactions. We see that the requirement (3.19) arises naturally in the course of the functional integration. We define the following function from the self-energy term:

$$g(x - x') = \exp\left\{ -\frac{i}{2} \int d^4y\, d^4y'\, \bar{J}_\mu(x, x'; y) D_1^{\mu\nu}(y - y') \bar{J}_\nu(x, x'; y') \right\} \tag{3.34}$$

g can be calculated explicitly, and for the Ansatz (3.21) this is done in Appendix A. Using g, we introduce the new functions a_{1g}, b_{1g} by defining a map $\mathbf{g}$: $a \to a_g$, $b \to b_g$ specified by the prescriptions

$$a_{1g}'(x) = g(x) a_1'(x) \tag{3.35}$$

$$b_{1g}(x) = g(x) b_1(x) \tag{3.36}$$

Here we use the notation $a'(x) = (d/dr)a(r)$, $r = -m^2x^2$. The uncertainty in a_{1g} due to the free integration constant is removed by noting that $g(0) = 1$ [this follows from condition (3.19)] and consequently requiring the same behavior for $a_{1g}(x)$ and $a_1(x)$ at $x \to 0$.

Now we may reverse the procedure indicated in equation (3.31) and reexponentiate the terms of the infinite sum under the remaining fermionic

integration, which, however cannot be done in a closed form. Proceeding in this way, we obtain

$$
\begin{aligned}
&\exp(i\Gamma_{II}[A, \Psi, \bar{\Psi}]) \\
&\quad = C \exp(i\Gamma_I^G[A]) \\
&\qquad \times \int D\psi\, D\bar{\psi} \exp\left\{ i \int d^4x\, [\bar{\eta}_I(x)\psi(x) + \bar{\psi}(x)\eta_I(x)] \right\} \\
&\qquad \times \exp\left\{ i \int d^4x\, d^4x'\, [\bar{\psi}(x) + \bar{\Psi}(x)] \exp\left[ie \int_x^{x'} dy_\mu\, A^\mu(y) \right] \right. \\
&\qquad \times S_{I(g)}^{-1}(x - x')[\psi(x') + \Psi(x')] \\
&\qquad - \frac{1}{2} \int d^4x\, d^4x'\, d^4z\, d^4z' \\
&\qquad \times \Big([\bar{\psi}(x) + \bar{\Psi}(x)] S_{I(g)}^{-1}(x - x')[\psi(x') + \Psi(x')] \\
&\qquad \times [\bar{\psi}(z) + \bar{\Psi}(z)] S_{I(g)}^{-1}(z - z')[\psi(z') + \Psi(z')] \\
&\qquad \times \left\{ \exp\left[-i \int d^4y\, d^4y'\, \bar{J}_\mu(x, x'; y) D_I^{\mu\nu}(y - y') \bar{J}_\nu(z, z'; y') \right] - 1 \right\} \Big) \\
&\qquad \left. + \cdots \right\}
\end{aligned}
\tag{3.37}
$$

with

$$
S_{I(g)}^{-1}(x - x') = [i\, \partial_x a_{Ig}(x - x') - m b_{Ig}(x - x')] \tag{3.38}
$$

In the last term of equation (3.37) we have already put $A_\mu = 0$ because we will only consider one-loop contributions [i.e., those stemming in equation (3.37) from the quadratic kernel of the fermion action in the presence of the arbitrary gauge potential A_μ] to the quadratic kernel of the gauge field action Γ_{II}^G.[14] In equation (3.37) the remaining fermionic integration is now done (in the sense of perturbation theory, and which after integration is formally summed up again). In performing the Gaussian integration [i.e. treating the last term (and all further terms) in equation (3.37) as a perturbation], for calculational simplicity we neglect the source terms (linear in Ψ, $\bar{\Psi}$; others are not of our present interest) that contain $[g(x) - 1]$ factors. For our

[14] As long as α is sufficiently small, higher loop contributions will only lead to small quantitative changes.

envisaged study, due to $g(0) = 1$ these source terms are irrelevant in the asymptotic UV region and in the asymptotic IR region they will lead to certain changes, which, however, are apparently small as long as α is sufficiently small. Now, without appealing to the eventual range of α, we simply understand this neglect as a certain further modification of the map $\tilde{f}$ but which preserves all important features (in particular, it does not lead to any change in the asymptotic UV region). So we obtain for equation (3.37) the following result[15]:

$$\begin{aligned}
&\exp(i\Gamma_{\mathrm{II}}[A, \Psi, \bar{\Psi}]) \\
&\quad = C \exp(i\Gamma_{\mathrm{I}}^{G}[A] + i\Delta\Gamma_{\mathrm{I}}^{G}[A]) \\
&\qquad \times \exp\Big\{ i \int d^4x\, d^4x'\, \bar{\Psi}(x) S_{\mathrm{I}(g)}^{-1}(x - x')\Psi(x') - \int d^4x\, d^4x'\, d^4z\, d^4z' \\
&\qquad \times \bar{\Psi}(x) S_{\mathrm{I}(g)}^{-1}(x - x') S_{\mathrm{I}(g)}(x' - z) S_{\mathrm{I}(g)}^{-1}(z - z')\Psi(z') \\
&\qquad \times \int d^4y\, d^4y'\, \bar{J}_{\mu}(x, x'; y) D^{\mu\nu}(y - y') \bar{J}_{\nu}(z, z'; y') + \cdots \Big\} \qquad (3.39)
\end{aligned}$$

Here $\Delta\Gamma_{\mathrm{I}}^{G}$ is defined by the equation

$$\exp(i\Delta\Gamma_{\mathrm{I}}^{G}[A]) = \int D\psi\, D\bar{\psi} \exp(i\Gamma_{\mathrm{I}}^{f}[A, \psi, \bar{\psi}]) \qquad (3.40)$$

However, as is clear from equation (3.37), for the present purpose in equation (3.14) a_{I}, b_{I} have to be replaced by $a_{\mathrm{I}g}$, $b_{\mathrm{I}g}$, respectively. Equation (3.39) provides us with those terms of the image Γ_{II} of Γ_{I} we need to know for our further investigation. So we may now proceed to apply the fixed-point condition to the kernel of the fermion action.

3.3.2. The Integral Equation for the Kernel of the Fermion Action

Considering $\Gamma_{\mathrm{II}}[0, \Psi, \bar{\Psi}] = \Gamma_{\mathrm{II}}^{f}[0, \Psi, \bar{\Psi}]$ and writing the quadratic terms as

$$\begin{aligned}
&\Gamma_{\mathrm{II}}^{f}[0, \Psi, \bar{\Psi}] \\
&\quad = \int d^4x\, d^4x'\, \bar{\Psi}(x)[ia_{\mathrm{II}}(x - x')\, \partial_{x'} - mb_{\mathrm{II}}(x - x')]\Psi(x') \qquad (3.41)
\end{aligned}$$

equation (3.39) provides us with expressions for a_{II}, b_{II}. Consequently, we

[15] We display only noninteraction terms of Γ_{II}, in which we are exclusively interested. Furthermore, on the r.h.s. only the term containing one photon propagator is shown.

may explicitly write down the fixed-point condition $a_{\mathrm{I}} = a_{\mathrm{II}}$, $b_{\mathrm{I}} = b_{\mathrm{II}}$. For convenience, we do it in terms of a_g, b_g, but any information obtained for these quantities can be translated into terms of a, b by means of relations (3.35), (3.36). The integral equation reads[16]

$$[g(x-z')-1]S_{(g)}^{-1}(x-z')$$
$$= -ig(x-z')\Big\{\int d^4x'\, d^4z\, S_{(g)}^{-1}(x-x')S_{(g)}(x'-z)S_{(g)}^{-1}(z-z')$$
$$\times \int d^4y\, d^4y'\, \bar{J}_\mu(x,x';y)D_{\mathrm{I}}^{\mu\nu}(y-y')\bar{J}_\nu(z,z';y') + \cdots\Big\} \quad (3.42)$$

Equation (3.42) represents two coupled integral equations for a_g, b_g and now needs to be solved. In general, this is a complicated task and we will restrict ourselves to the solution of equation (3.42) in the asymptotic UV [i.e., $-m^2(x-z')^2 \to 0$] and IR [i.e., $-m^2(x-z')^2 \to \infty$] regions, respectively.[17] Before studying these cases let us mention that equation (3.42) has an exact but trivial solution, namely

$$a_g(x) = a(x) \equiv 0 \quad (3.43)$$
$$b_g(x) = b(x) = \tilde{b}(\infty)\delta^{(4)}(x) \quad (3.44)$$

where $\tilde{b}(\infty)$ is some arbitrary real constant. Of course, this solution corresponds to the noninteracting case where the gauge and fermion sectors are decoupled and it is therefore not very interesting. However, in view of (3.15) and (3.16), in the following we will search for the interacting solution of equation (3.42) as the sum of the trivial solution (3.43), (3.44) and some additional nontrivial contribution. As already mentioned, it seems to be rather complicated to find a nontrivial and exact solution of equation (3.42), but it appears possible to analyze it merely exactly, at least in the asymptotic UV region and for small α to leading order in the IR region solely based on those terms explicitly displayed in it. First we will turn to the asymptotic UV region.

3.3.2.1. Solving the Integral Equation in the Asymptotic UV Region. Exploring equation (3.42), one soon recognizes that to find a solution correct in the asymptotic UV region one needs to assume that the photon propagator $D_{\mathrm{I}}^{\mu\nu}(x)$ is finite in the coincidence limit $x \to 0$. Consequently, the photon propagator written as[18]

[16] Note that the fixed-point condition has been multiplied by $g(x-z')$ also.
[17] We always have the Euclidean region in mind, of course.
[18] An appropriate gauge-fixing term Γ_{gf} has been added to the gauge field action Γ_{I}^{G}, i.e., we have chosen $\tilde{n}_\mu(p) = ip_\mu\, \tilde{d}(p)^{1/2}$.

$$D^{\mu\nu}(x) = -\int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{p^2 + i\epsilon} \frac{1}{\tilde{d}(p)} \left[g^{\mu\nu} - (1 - \lambda) \frac{p^\mu p^\nu}{p^2 + i\epsilon} \right] \tag{3.45}$$

reads in the coincidence limit

$$D^{\mu\nu}(0) = ig^{\mu\nu} \frac{3 + \lambda}{4} K_A m^2 \tag{3.46}$$

$$K_A = \frac{1}{4\pi^2} \int_0^\infty ds \, \frac{1}{\tilde{d}(s)}$$

where K_A is some finite, real constant.

The analysis of the integral equation (3.42) in the asymptotic UV region now starts by replacing the photon propagator (3.45) by its leading short-distance term (3.46). Consequently, the current–current interaction then reads in the short-distance limit

$$\int d^4y \, d^4y' \, \bar{J}_\mu(x, x'; y) D^{\mu\nu}(y - y') \bar{J}_\nu(z, z'; y')$$

$$= i\alpha\pi(3 + \lambda) K_A m^2 (x - x')(z - z') + \cdots \tag{3.47}$$

and the function g has the short-distance behavior

$$g(x) = 1 + \frac{\alpha\pi}{2} (3 + \lambda) K_A m^2 x^2 + \cdots \tag{3.48}$$

The leading short-distance terms (3.47), (3.48) have to be inserted into the integral equation (3.42), yielding

$$\frac{1}{2} (x - z')^2 S_{(g)}^{-1}(x - z') = \int d^4x' \, d^4z \, S_{(g)}^{-1}(x - x') S_{(g)}(x' - z)$$

$$\times S_{(g)}^{-1}(z - z') \, (x - x')(z - z') + \cdots \tag{3.49}$$

Here, certain constants have been divided out. For convenience we will now further consider the above integral equation in momentum space. For this purpose we translate coordinate difference factors that occur [i.e., $(x - z')^2$, $(x - x')(z - z')$] into momentum-space derivatives. Having this in mind, one may convince oneself that to leading order, terms (indicated by dots . . .) containing more than just one photon propagator and which are not all coupled to a closed fermion loop do not contribute because they are related to a higher number of derivatives in momentum space (and those terms then are falling off faster in the UV—i.e., high-momentum—region). Effectively, the terms displayed on the r.h.s. of equation (3.49) are only modified by diagrams where all photon propagators are coupled to closed fermion loops. However,

these closed fermion loops can always be summed up to give an effective (modified) photon propagator. As long as its coincidence limit remains finite, equation (3.49) applies. So, once the assumption (3.20) is accepted, the UV analysis can be done exactly. In addition, already from equation (3.49) we recognize that the leading UV term of the solution we are in search of is independent of the coupling constant α as well as of the structure of the gauge field action [beyond condition (3.20)] determining the constant K_A, which will give the UV behavior a universal character.

Equation (3.49) now reads in momentum space (the subscript g is omitted for the moment)

$$\begin{aligned} &\not{p}[s\tilde{a}'' + 3\tilde{a}'] + m[s\tilde{b}'' + 2\tilde{b}'] \\ &= \frac{2}{s\tilde{a}^2 + \tilde{b}^2} \\ &\quad \times \{\not{p}[s^2\tilde{a}(\tilde{a}')^2 + s\tilde{a}^2\tilde{a}' + 2s\tilde{a}'\tilde{b}\tilde{b}' - s\tilde{a}(\tilde{b}')^2 - \tfrac{1}{2}\tilde{a}^3 + \tilde{a}\tilde{b}\tilde{b}'] \\ &\quad + m[2s^2\tilde{a}\tilde{a}'\tilde{b}' - s^2(\tilde{a}')^2\tilde{b} + s\tilde{a}^2\tilde{b}' - s\tilde{a}\tilde{a}'\tilde{b} + s\tilde{b}(\tilde{b}')^2 - \tilde{a}^2\tilde{b}]\} \\ &\quad + \cdots \end{aligned} \tag{3.50}$$

Here, the notation is $\tilde{a} = \tilde{a}(s)$, $\tilde{a}' = (d/ds)\tilde{a}$, $s = -p^2/m^2$. We will now solve the two coupled differential equations represented by equation (3.50) in the asymptotic UV region $s \to \infty$. Our Ansatz, in accordance with conditions (3.15), (3.16), will be $\tilde{a} = \tilde{a}_s$, $\tilde{b} = \tilde{b}(\infty) + \tilde{b}_s$, where $\tilde{a}_s$, $\tilde{b}_s$ are assumed to vanish in a power-like way in leading order for $s \to \infty$. Neglecting all clearly nonleading terms, we find that the two coupled differential equations yielded by equation (3.50) then read[19]

$$\tfrac{1}{2}s\tilde{a}_s'' + \tfrac{3}{2}\tilde{a}_s' \overset{s\to\infty}{=} s^2\tilde{a}_s(\tilde{a}_s')^2 + s\tilde{a}_s^2\tilde{a}_s' - \tfrac{1}{2}\tilde{a}_s^3 + [2s\tilde{a}_s' + \tilde{a}_s]\tilde{b}_s' + \cdots \tag{3.51}$$

$$\tfrac{1}{2}s\tilde{b}_s'' + \tilde{b}_s' \overset{s\to\infty}{=} -s^2(\tilde{a}_s')^2 - s\tilde{a}_s\tilde{a}_s' - \tilde{a}_s^2 + \cdots \tag{3.52}$$

Let us first discuss equation (3.51) and its consequences for the asymptotic UV behavior of $\tilde{b}_s$. As long as the term on the l.h.s. of equation (3.51) does not vanish to leading order we are forced to conclude that[20]

$$\tilde{b}_s' \overset{s\to\infty}{\sim} 1/s, \quad \text{i.e.,} \quad \tilde{b}_s \overset{s\to\infty}{\sim} \ln s$$

However, such a behavior is in conflict with gauge invariance because it is

[19] Note that also a temporary transition $\tilde{a}_s \to \tilde{b}(\infty)\tilde{a}_s$, $\tilde{b}_s \to \tilde{b}(\infty)\tilde{b}_s$ has been applied and then the factor $2\tilde{b}(\infty)$ has been divided out of the equations below.

[20] Of course, one could also try the assumption that the term in front of $\tilde{b}_s'$ vanishes (i.e., $\tilde{a}_s \overset{s\to\infty}{\sim} s^{-1/2}$); however equation (3.52) immediately leads to the same result.

not in line with condition (3.16). So, we are led to conclude that the l.h.s. of equation (3.51) should vanish to leading order, consequently the following must hold ($C_{\tilde{a}}$ is some constant):

$$\tilde{a}_s \overset{s\to\infty}{=} \frac{C_{\tilde{a}}}{s^2} + \cdots \tag{3.53}$$

This information is sufficient to determine the leading behavior of $\tilde{b}_s$ from equation (3.52), and we find

$$\tilde{b}_s \overset{s\to\infty}{=} -\frac{C_{\tilde{a}}^2}{s^3} + \cdots \tag{3.54}$$

We may now come back to equation (3.51) and determine the next-to-leading term of $\tilde{a}_s$. Writing $\tilde{a}_s$ without any loss of generality as

$$\tilde{a}_s = \frac{C_{\tilde{a}}}{s^2}\,\tilde{v}(s), \qquad \tilde{v}(\infty) = 1 \tag{3.55}$$

and taking into account (3.53), (3.54), we find that equation (3.51) then reads[21]

$$\frac{C_{\tilde{a}}}{2}\frac{1}{s^2}\,[s\tilde{v}'' - \tilde{v}'] \overset{s\to\infty}{=} -\frac{15}{2}\frac{C_{\tilde{a}}^3}{s^6} + \cdots \tag{3.56}$$

and we find

$$\tilde{v}(s) \overset{s\to\infty}{=} 1 - \frac{C_{\tilde{a}}^2}{s^3} + \cdots \tag{3.57}$$

Summarizing the above results, one can say that equation (3.42) admits a (unique) solution respecting conditions (3.15), (3.16). It behaves in the asymptotic UV region as follows:

$$\tilde{a}_g(s) \overset{s\to\infty}{=} \frac{C_{\tilde{a}}}{s^2}\,\tilde{b}(\infty)\left[1 - \frac{C_{\tilde{a}}^2}{s^3} + \cdots\right] \tag{3.58}$$

$$\tilde{b}_g(s) \overset{s\to\infty}{=} \tilde{b}(\infty)\left[1 - \frac{C_{\tilde{a}}^2}{s^3} + \cdots\right] \tag{3.59}$$

[21] To be more precise, the vanishing of the leading term on the l.h.s. of equation (3.56) [equation (3.55) inserted] rests on the relation [$\hat{p} = (-p_0, \mathbf{p})$]

$${}_p\Box\,\frac{\not{p}}{[p^2]^2} = i\cdot 2\pi^2\,\partial_{\not{p}}\delta^{(4)}(p)$$

accompanied by certain reasonable assumptions about $\tilde{a}_s(s \to 0)$ (i.e., $\tilde{v} \sim s^2$, $s \to 0$; or even some weaker condition).

Most important, in a qualitative respect this asymptotic UV behavior is independent of the coupling constant α and of any specific details of the photon propagator structure beyond condition (3.20). Furthermore, due to $g(0) = 1$ [cf. equation (3.34)], $\tilde{a}$, $\tilde{b}$ exhibit the same leading UV behavior as $\tilde{a}_g$, $\tilde{b}_g$. We will discuss the consequences of the above results in Sections 3.3.3 and 4. In the next subsection we will study equation (3.42) in the asymptotic IR region.

3.3.2.2. Solving the Integral Equation in the Asymptotic IR Region. For the IR analysis of the integral equation (3.42) we need to apply our Ansatz (3.21) to the photon propagator.[22] Consequently, the current–current interaction reads in the long-distance limit to leading order[23]

$$\int d^4y\, d^4y'\, \bar{J}_\mu(x, x'; y) D_{\mathrm{f}}^{\mu\nu}(y - y') \bar{J}_\nu(z, z'; y')$$

$$= i \frac{\alpha}{\pi} \left\{ \frac{1 + \lambda}{2} \frac{(x - x')(z - z')}{(x - z')^2} + (1 - \lambda) \frac{(x - x')(x - z')\,(x - z')(z - z')}{[(x - z')^2]^2} \right\} + \cdots \tag{3.60}$$

Here, $(x - x')^2$, $(z - z')^2$ are understood to be small compared with $(x - z')^2$.[24] The function g has the long-distance behavior (we give it here just for Euclidean space, for the full expression and its derivation see Appendix A)

$$g(x_{\mathrm{E}}) = C_g\, (m^2 x_{\mathrm{E}}^2)^{\alpha(3-\lambda)/4\pi} \left[\exp\left(-\frac{\alpha}{2\sqrt{\beta}}\, m |x_{\mathrm{E}}| \right) \right] [1 + \cdots] \tag{3.61}$$

$$C_g = (4\beta)^{-\alpha(3-\lambda)/4\pi} \exp\left\{ \frac{\alpha}{4\pi} [(3 + \lambda) + 2(3 - \lambda)\gamma] \right\}$$

Note that equation (3.61) contains the Bloch–Nordsieck contribution (Kernemann and Stefanis, 1989; Karanikas *et al.*, 1992; and references therein) exhibiting a power-like behavior with the well-known exponent $\alpha(3$

[22] To obtain this propagator a gauge-fixing term Γ_{gf} with $\tilde{n}_\mu = ip_\mu\, \tilde{d}(p)^{1/2}$ has been added to the gauge field action $\Gamma_{\mathrm{f}}^{\mathrm{G}}$.

[23] Of course, it is not specifically related to the Ansatz (3.21); only next-to-leading terms will be influenced.

[24] We always have in mind the region $-m^2(x - z')^2 \to \infty$. More precisely, for any large but fixed value of $(x - z')^2$ contributions from integration regions in the integral equation (3.42) where $(x - x')^2$, $(z - z')^2$ are not small compared to $(x - x')^2$ can be expected to be small due to the expected decay of a_g, b_g there. Furthermore, terms containing higher powers of $1/(x - z')^2$ are suppressed in the asymptotic IR region whatever their coefficient numerically might be.

$-\lambda)/4\pi$. It appears justified to assume that the leading IR behavior displayed in equations (3.60), (3.61) will depend on additional terms to be introduced in the Ansatz (3.21) in order also to satisfy condition (3.20) only very weakly. For the purpose of calculational simplicity those terms can therefore be safely disregarded.

We may now insert equations (3.60), (3.61) into the integral equation (3.42). Having in mind IR analysis in Euclidean space, on the l.h.s. of equation (3.42) we replace the factor $[1 - g(x - z')]$ simply by 1 because this is the leading contribution due to the exponential decay (i.e., oscillation in Minkowski space) of $g(x_E)$ for $m^2 x_E^2 \to \infty$. Furthermore, coordinate difference factors [i.e., $(x - x')_\mu$, $(z - z')_\nu$] occurring on the r.h.s. of equation (3.60) are translated into momentum-space derivatives acting on the Fourier transform of the kernel S^{-1} of the fermion action. So, equation (3.42) reads now

$$
\begin{aligned}
&[i\,\partial\!\!\!/_x a_g(x - z') - m b_g(x - z')] \\
&= \frac{\alpha}{\pi}\, g(x - z') \left[\frac{1 + \lambda}{2} \frac{g^{\mu\nu}}{(x - z')^2} + (1 - \lambda) \frac{(x - z')^\mu (x - z')^\nu}{[(x - z')^2]^2} \right] \\
&\quad \times \int \frac{d^4p}{(2\pi)^4}\, e^{ip(x - z')} \{ {}_p\partial_\mu [p\!\!\!/ \tilde{a}_g(p) + m \tilde{b}_g(p)] \} \\
&\quad \times \frac{p\!\!\!/ \tilde{a}_g(p) - m \tilde{b}_g(p)}{p^2 \tilde{a}_g^2(p) - m^2 \tilde{b}_g^2(p) + i\epsilon} \{ {}_p\partial_\nu [p\!\!\!/ \tilde{a}_g(p) + m \tilde{b}_g(p)] \} \\
&\quad + \cdots
\end{aligned}
\tag{3.62}
$$

Concerning the contribution of terms containing more than just one photon propagator (indicated by dots ...), the following comments are due. Most of those terms will finally yield higher powers of $1/(x - z')^2$ at least and these can therefore be neglected in the asymptotic IR region. However, one should expect that terms also occur which are of the same order as the one-loop term given above. Such terms should be expected to contribute only weakly numerically as long as α is sufficiently small, because each additional photon propagator is accompanied by an additional factor of α. This argument is what is left within the present approximative approach of the line of reasoning applied in standard QED perturbation theory. Of course, the belief based on this reasoning may turn out to be wrong because of nonperturbative mechanisms which are not easily seen at the present stage of the investigation. Anyway, in the region where α is of order 1, terms containing more than just one photon propagator can no longer be neglected in principle. However, for the purpose of the present model calculation (without appealing to the eventual range of α) we simply ignore all terms containing more than just one photon propagator also in the region where α is not small.

To determine the IR tail of a_g, b_g [i.e. the l.h.s. of equation (3.62)] it remains to find the leading long-distance contribution of the Fourier integral on the r.h.s. of equation (3.62). To proceed further we would preferably need to know the analytic structure of the integrand, in particular that of the denominator. We do not have any reliable information on this, but it appears reasonable to assume that the integrand has a simple pole at some $p_0 = \pm \sqrt{\mathbf{p}^2 - s_0 m^2}$ with

$$s_0 = -\frac{\tilde{b}_g(s_0)^2}{\tilde{a}_g(s_0)^2} \qquad (s_0 < 0) \tag{3.63}$$

and that just this pole determines the leading long-distance behavior of the Fourier integral. Consequently, we may exploit the residue of this pole, and the leading long-distance contribution of the Fourier integral is simply given by the product of the nominator of its integrand [appropriately treated by considering p_κ factors occurring as configuration space derivatives acting on equation (3.64)] taken at $p^2/m^2 = -s_0$ and the leading long-distance term of

$$\frac{1}{\tilde{a}_g(s_0)^2} \int \frac{d^4p}{(2\pi)^4} \frac{e^{ip(x-z')}}{p^2 + s_0 m^2 + i\epsilon} \tag{3.64}$$

The explicit calculation is now straightforward but somewhat tedious. In performing the calculation we always keep track of those terms contributing in the long-distance region to leading order only. In particular, the leading long-distance term of equation (3.64) is read off from the relation (written for Euclidean space here)

$$\int \frac{d^4p_E}{(2\pi)^4} \frac{e^{ip_E x_E}}{p_E^2 + m^2} = \frac{m}{4\pi^2 |x_E|} K_1(m|x_E|)$$

$$\overset{m^2 x_E^2 >> 1}{=} \frac{\sqrt{m}}{2(2\pi)^{3/2} |x_E|^{3/2}} e^{-m|x_E|}[1 + \cdots] \tag{3.65}$$

The result obtained this way for the IR tail of a_g, b_g is then (we give this and all further results for Euclidean space)

$$a_g(x_E) \overset{m^2 x_E^2 \to \infty}{=} m^4 \frac{\alpha C_g G}{(2\pi)^{5/2}} \frac{(-s_0)^{3/4} \tilde{a}_g(s_0)}{\sqrt{-s_0} + \alpha/2\sqrt{\beta}} (m|x_E|)^{-7/2 + \alpha(3-\lambda)/2\pi}$$

$$\times \{\exp[-(\sqrt{-s_0} + \alpha/2\sqrt{\beta}) m|x_E|]\}[1 + \cdots] \tag{3.66}$$

$$b_g(x_E) \overset{m^2 x_E^2 \to \infty}{=} \frac{H}{G} (\sqrt{-s_0} + \alpha/2\sqrt{\beta}) a_g(m^2 x_E^2 \to \infty) \tag{3.67}$$

$$G = -\frac{3}{2}(1 + \lambda) + 2(3 - \lambda)\left[s_0 \frac{\tilde{a}_g'(s_0)}{\tilde{a}_g(s_0)} + \sqrt{-s_0} \frac{\tilde{b}_g'(s_0)}{\tilde{a}_g(s_0)} + \frac{1}{2} \right]^2 \tag{3.68}$$

$$H = -3(1 + \lambda) - G \tag{3.69}$$

Equation (3.68) provides us with an implicit expression for G only because in view of equations (3.67), (3.69) its r.h.s. also depends on G via the term $\tilde{b}_g'(s_0)/\tilde{a}_g(s_0)$. Therefore equation (3.68) represents a cubic equation for the value of G which has always at least one (real) solution. From equation (3.68) one recognizes that G is an RG-invariant quantity, i.e., it is invariant against (finite) mass and (fermion) wave function renormalizations (we will discuss the normalization issue below).

Taking into account the definitions (3.35), (3.36), we find from equations (3.66), (3.67) the IR tail of a, b,

$$a(x_E) \overset{m^2x_E^2\to\infty}{=} m^4 \frac{\alpha G}{(2\pi)^{5/2}} (-s_0)^{1/4} \tilde{a}_g(s_0)(m|x_E|)^{-7/2} \times [\exp(-\sqrt{-s_0}\, m|x_E|)][1 + \cdots] \tag{3.70}$$

$$b(x_E) \overset{m^2x_E^2\to\infty}{=} \frac{H}{G} \sqrt{-s_0}\, a(m^2x_E^2 \to \infty) \tag{3.71}$$

From the above equations we see that the IR tails of a, b agree qualitatively (the same is true for a_g, b_g).

After having obtained the functional dependence of the kernel of the fermion action in the asymptotic IR region, we still need to fix the arbitrary constants involved (in particular, this will require the discussion of the normalization issue not touched on so far). For this purpose we have to calculate the Fourier transforms of a_g, b_g and those of a, b, the latter of which are determined by the solution of the integral equation (3.42) via equations (3.35), (3.36). It appears reasonable to represent these Fourier transforms in the low-s region,[25] which is appropriate for the normalization purposes, by the sum of the Fourier transforms of the trivial solution (3.43), (3.44) and the Fourier transforms $\tilde{a}_{sg}$, $\tilde{b}_{sg}$, $\tilde{a}_s$, $\tilde{b}_s$ of the IR tails of a_g, b_g and a, b given in equations (3.66), (3.67) and (3.70), (3.71), respectively. So we simply extend the long-distance representations (3.66), (3.67), (3.70), and (3.71) to the whole configuration space and expect that this procedure will give reasonable results in the low-s region at least.

The following formula applies to the calculation of the Fourier transforms (Prudnikov *et al.*, 1990–1992):

[25] In the following we will deliberately leave open the precise meaning of this term. We will return to the issue in Section 3.3.3.

$$\int d^4x_E\, e^{-ip_E x_E}(x_E^2)^\kappa e^{-\rho|x_E|}$$

$$= -\frac{4\pi^2\Gamma(4+2\kappa)}{|p_E|(\rho^2+p_E^2)^{3/2+\kappa}} P_{2(1+\kappa)}^{-1}\left(\frac{\rho}{\sqrt{\rho^2+p_E^2}}\right)$$

$$= \frac{4\pi^2\Gamma(3+2\kappa)}{p_E^2(\rho^2+p_E^2)^{3/2+\kappa}}$$

$$\times\left[\sqrt{\rho^2+p_E^2}P_{1+2\kappa}\left(\frac{\rho}{\sqrt{\rho^2+p_E^2}}\right) - \rho P_{2(1+\kappa)}\left(\frac{\rho}{\sqrt{\rho^2+p_E^2}}\right)\right],$$

$$\mathrm{Re}\,\rho > 0, \quad \mathrm{Re}\,\kappa > -2 \tag{3.72}$$

Having in mind continuation to Minkowski space, note that (more precisely) the condition $|\mathrm{Im}\,|p_E|| < \mathrm{Re}\,\rho$ is to be respected. Although it is less compact, in the following we will always exploit the lower representation of equation (3.72) because we find it more convenient for an eventual transition back to Minkowski space. For simplifying some of the formulas below we define the function

$$T(s,\kappa) = (1+s)^{(1-2\kappa)/4}[\sqrt{1+s}P_{-5/2+\kappa}((1+s)^{-1/2}) - P_{-3/2+\kappa}((1+s)^{-1/2})] \tag{3.73}$$

For $\tilde{a}_g$ given in the low-s region as the Fourier transform of equation (3.66), we obtain the following result:

$$\tilde{a}_g(s) = \frac{\alpha C_g G}{\sqrt{2\pi}s}\Gamma\left(-\frac{1}{2}+\alpha\frac{3-\lambda}{2\pi}\right)(-s_0)^{3/4}\tilde{a}_g(s_0)$$

$$\times(\sqrt{-s_0}+\alpha/2\sqrt{\beta})^{1/2-\alpha(3-\lambda)/2\pi}$$

$$\times T(s(\sqrt{-s_0}+\alpha/2\sqrt{\beta})^2, \alpha(3-\lambda)/2\pi) \tag{3.74}$$

By specifying $s = s_0$ (this corresponds to an analytic continuation to Minkowski space), we find that the above equation leads to a consistency equation [the value of $\tilde{a}_g(s_0)$ drops out] yielding a first relation among the parameters of the IR solution. It reads

$$1 = -\alpha^{1-\alpha(3-\lambda)/2\pi}\frac{G}{\sqrt{2\pi}}\Gamma\left(-\frac{1}{2}+\alpha\frac{3-\lambda}{2\pi}\right)$$

$$\times\exp\left\{\frac{\alpha}{4\pi}[(3+\lambda)+2(3-\lambda)\gamma]\right\} w^{-1/2}(1+w)^{1/2-\alpha(3-\lambda)/2\pi}$$

$$\times T(-w^2(1+w)^{-2}, \alpha(3-\lambda)/2\pi) \tag{3.75}$$

$$w = \frac{2}{\alpha}\sqrt{-s_0\beta} \tag{3.76}$$

Here, G is understood as a function of w and α (and λ). It is given as a solution of the following cubic equation derived from equation (3.68):

$$\begin{aligned} G^3 &+ \left\{\frac{3}{2}(1+\lambda) - 2(3-\lambda)\left[\left(2+\frac{1}{w}\right)L(w,\alpha) + \frac{1}{2}\right]^2\right\}G^2 \\ &- 12(3-\lambda)(1+\lambda)\left(1+\frac{1}{w}\right)\left[\left(2+\frac{1}{w}\right)L(w,\alpha) + \frac{1}{2}\right]L(w,\alpha)G \\ &- 18(3-\lambda)(1+\lambda)^2\left(1+\frac{1}{w}\right)^2 L(w,\alpha)^2 = 0, \qquad L(w,\alpha) = s_0\frac{\tilde{a}_g'(s_0)}{\tilde{a}_g(s_0)} \end{aligned} \tag{3.77}$$

To obtain this cubic equation we have used the relation

$$\tilde{b}_g(s) = -\sqrt{-s_0}\left(1+\frac{1}{w}\right)\left[1+\frac{3(1+\lambda)}{G}\right]\tilde{a}_g(s) + \tilde{b}(\infty) \tag{3.78}$$

based on equations (3.66), (3.67) and therefore valid in the low-s region only. We see that solutions G of equation (3.77) are functions of w and α, while solutions w of equation (3.75) exclusively depend on α (and on λ, in principle, if for conceptual reasons we were not to set it to zero as outlined in Section 3.1). Clearly they do not depend on $\tilde{b}(\infty)$. Although numerically the discriminant of equation (3.77) always turns out to be negative in the relevant domain, only one of the three real solutions of equation (3.77) then proves appropriate to find a solution of equation (3.75). In general, solutions G and $w(\alpha)$ of the above equations can only be found numerically (for a plot of numerical results see Figs. 1 and 2). However, for sufficiently small α ($\alpha << 1$), $w(\alpha)$ turns out to be large, $w(\alpha) >> 1$, and equation (3.75) admits an analytical solution in this region. This asymptotic solution will be studied now.

We investigate the case $\alpha << 1$ [we assume that the solution $w(\alpha)$ in this region will be much larger than one]. Let us start with the following asymptotic representation (Gradshteyn and Ryzhik, 1972):

$$\begin{aligned} &z^{-1/2+\kappa}[z^{-1}P_{-5/2+\kappa}(z) - P_{-3/2+\kappa}(z)] \\ &\quad = \left(\frac{1}{2}-\kappa\right)\frac{\Gamma(1-\kappa)}{\Gamma(\frac{5}{2}-\kappa)}\frac{2^{1/2-\kappa}}{\sqrt{\pi}} \\ &\qquad \times\left[1 - \frac{1}{4\kappa z^2}\left(\frac{1}{2}-\kappa\right)\left(\frac{3}{2}-\kappa\right)\right. \end{aligned}$$

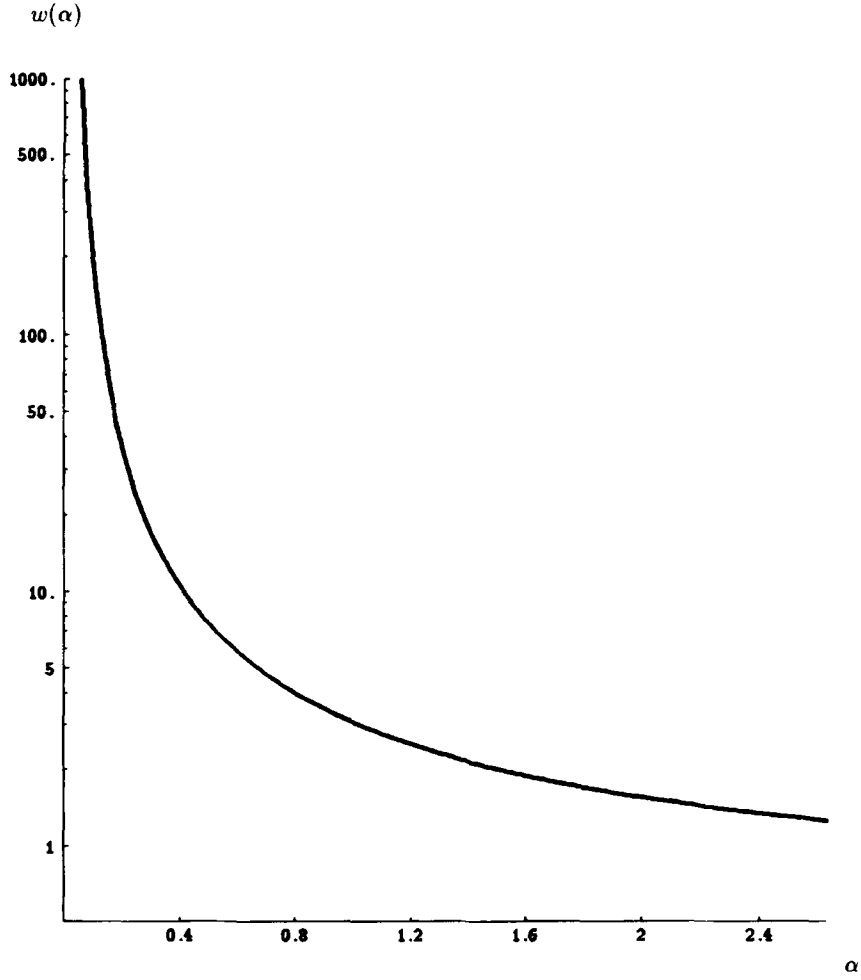

Fig. 1. Solution w of equation (3.75) as a function of α.

$$\times \left(\frac{(2z)^{2\kappa}}{1-\kappa} \frac{\Gamma(\frac{1}{2}-\kappa)}{\Gamma(\frac{1}{2}+\kappa)} \frac{\Gamma(1+\kappa)}{\Gamma(1-\kappa)} - 1 \right) + O(z^{-2(2-\kappa)}) \Bigg],$$

$$\kappa > 0, \quad |z| >> 1 \tag{3.79}$$

$$= \frac{2\sqrt{2}}{3\pi} \left[1 - \frac{3}{16} z^{-2}[2 \ln 8z + 1] + O(z^{-4} \ln z) \right],$$

$$\kappa = 0, \quad |z| >> 1 \tag{3.80}$$

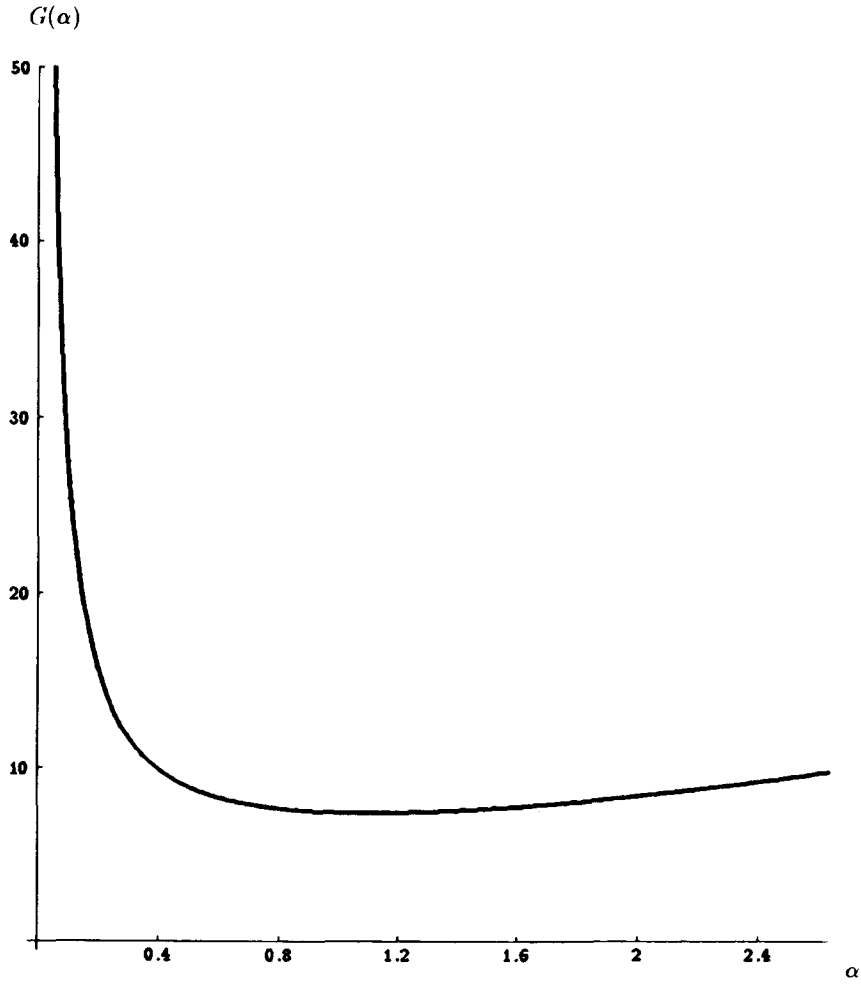

Fig. 2. Solution G of equation (3.83) [with w as solution of equation (3.75) inserted] as a function of α.

Then, from equation (3.75) one finds [here $\ln w(\alpha)$ is thought to grow for small α like $\alpha^{-1/2}$ at most]

$$G = \frac{3\pi}{4\alpha}\left\{1 - \frac{\alpha}{4\pi}\left[(3+\lambda) + 2(3-\lambda)\left(\frac{8}{3} - \ln\left[\frac{2^5 w(\alpha)}{\alpha}\right]\right)\right]\right.$$
$$\left. + \frac{1}{2}\left(\frac{\alpha(3-\lambda)}{2\pi}\right)^2 \ln w(\alpha)\,\ln[\alpha^4 w(\alpha)] + O(\alpha^{3/2})\right\} \tag{3.81}$$

Taking into account (3.81), we can then insert equation (3.78) on the r.h.s. of equation (3.68) and equation (3.81) on its l.h.s. The solution of the resulting equation for $w(\alpha)$ is now straightforward. One finds for small α

$$w(\alpha) = \frac{1}{32}\exp\left\{\frac{2}{3}\sqrt{\frac{2\pi}{\alpha(1-\lambda/3)}} + 4 + \sqrt{\frac{\alpha(1-\lambda/3)}{2\pi}}\ln\alpha \right.$$
$$\left. - \frac{1}{6}\sqrt{\frac{\alpha(1-\lambda/3)}{2\pi}}\left[\frac{59}{3} + \frac{38\lambda}{(3-\lambda)}\right] + O(\alpha)\right\}, \qquad \alpha << 1 \tag{3.82}$$

Note that higher loop contributions that may possibly be taken into account in the integral equation (3.42) will influence the above result via the last term in the exponent only. To see this, simply replace α by $\alpha[1 + O(\alpha)]$ in the first term in the exponent. Finally, using (3.82), one finds from equation (3.81) the following expression for $G(\alpha)$:

$$G(\alpha) = \frac{3\pi}{4\alpha}\left\{1 + 2\sqrt{\frac{\alpha(1-\lambda/3)}{2\pi}} + \frac{\alpha(3-\lambda)}{2\pi}\ln\alpha \right.$$
$$\left. + \frac{\alpha(9-5\lambda)}{4\pi} + 15\left(\frac{\alpha(1-\lambda/3)}{2\pi}\right)^{3/2}\ln\alpha + O(\alpha^{3/2})\right\}, \qquad \alpha << 1 \tag{3.83}$$

The next task is to find the solution s_0 of equation (3.63). However, any solution s_0 can only sensibly be related to physics if the mass normalization to be used is specified. So, before attempting to find s_0, we discuss the normalization issue in somewhat greater detail. Let us assume we had determined s_0. Then, whatever normalization of $\tilde{a}_g(s_0)$ is applied, equation (3.63) yields the value of $\tilde{b}_g(s_0)$, and in our specific case the value of $\tilde{b}(\infty)$, because equation (3.67) is not independent of equation (3.66). Now, let a certain function $\hat{g} = \hat{g}(-m^2x^2)$ with $\hat{g}(0) = 1$ define a map $\hat{\mathbf{g}}$: $a_g \to a_{g\hat{g}}$, $b_b \to b_{g\hat{g}}$ by applying the prescriptions (3.35), (3.36) to $\hat{g}$. Considering the equation

$$s_1 = -\frac{\tilde{b}_{g\hat{g}}(s_1)^2}{\tilde{a}_{g\hat{g}}(s_1)^2} \qquad (s_1 < 0) \tag{3.84}$$

we see that the map $\hat{\mathbf{g}}$ obviously induces a map $\hat{\mathbf{g}}_s$: $s_0 \to s_1$. If $\hat{g} \equiv 1$, $\hat{\mathbf{g}}$ and $\hat{\mathbf{g}}_s$ are the identity maps. If we specifically choose $\hat{g} = g^{-1}$, then $\hat{\mathbf{g}}$ is the inverse of $\mathbf{g}$ and $a_{g\hat{g}} = a$, $b_{g\hat{g}} = b$ [cf. equations (3.35), (3.36)]. However, a, b are related to physics and we would like to formulate normalization conditions in terms of them. We naturally prefer to impose standard normalization

conditions on $\tilde{a}$, $\tilde{b}$ (i.e., mass shell normalization at the physical electron mass m):

$$\tilde{a}(s_1 = -1) = \pm\tilde{b}(s_1 = -1) = N_2^{-1} = 1 \tag{3.85}$$

In other words, we of course require that the fermion propagator derived from the effective action we are in search of has a pole related to the physical electron mass m. In equation (3.85) N_2 is the (fermion) wave function normalization constant.[26] Note that it is always possible to choose $s_1 = -1$ because in our setup there exists a scaling symmetry $m \to \tau m$ ($s \to s/\tau^2$), $\beta \to \tau^2\beta$, $b \to b/\tau$ for any nonzero real parameter τ [RG invariance against (finite) mass renormalizations]. Consequently, we now apply the inverse map $\hat{\mathbf{g}}_s^{-1}$: $s_1 \to s_0$ to determine s_0. Taking into account [cf. equations (3.70), (3.71)]

$$\tilde{b}(s) = -\sqrt{-s_0}\left[1 + \frac{3(1+\lambda)}{G}\right]\tilde{a}(s) + \tilde{b}(\infty) \tag{3.86}$$

(valid in the low-s region) and the low-s result for the Fourier transform of a,

$$\tilde{a}(s) = \sqrt{2}\alpha G(\alpha)\tilde{a}(s_0)(s_0/s)T(-s/s_0, 0) \tag{3.87}$$

we conveniently calculate for the s_1 pole via $\sqrt{-s_1}\tilde{a}(s_1) = \pm\tilde{b}(s_1)$ the value of the RG-invariant quantity $\tilde{b}(\infty)/[\sqrt{-s_0}\tilde{a}_g(s_0)]$ [i.e., the value of the RG-variant quantity $\tilde{b}(\infty)$ expressed in terms of s_0 and $\tilde{a}_g(s_0)$]. We find

$$\frac{\tilde{b}(\infty)}{\sqrt{-s_0}\tilde{a}_g(s_0)} = \sqrt{2u}\alpha G(\alpha)\left\{\pm 1 + \sqrt{u}\left[1 + \frac{3(1+\lambda)}{G(\alpha)}\right]\right\}$$

$$\times T(-u^{-1}, 0), \qquad u = \frac{s_0}{s_1} \tag{3.88}$$

The same quantity can now be found from the s_0 pole via $\sqrt{-s_0}\tilde{a}_g(s_0) = \tilde{b}_g(s_0)$,[27] and both values have to agree, of course, which provides us with an equation for s_0 measured in units of s_1, which in our case ($s_1 = -1$) is related to the physical electron mass m. The equation reads

$$1 + \left(1 + \frac{1}{w(\alpha)}\right)\left[1 + \frac{3(1+\lambda)}{G(\alpha)}\right]$$

$$= \sqrt{-2s_0}\alpha G(\alpha)\left\{\pm 1 + \sqrt{-s_0}\left[1 + \frac{3(1+\lambda)}{G(\alpha)}\right]\right\}T(s_0^{-1}, 0) \tag{3.89}$$

[26] A (finite) wave function renormalization corresponds to a change in N_2.

[27] We omit the other root, $\sqrt{-s_0}\,\tilde{a}_g(s_0) = -\tilde{b}_g(s_0)$, because one does not find any solution s_0 in this case.

Again, in general, solutions $s_0(\alpha)$ of this equation can only be studied numerically (see Fig. 3). However, for very small α ($\alpha << 1$), where s_0 is very close to -1, it can also be investigated analytically and one finds [choose the upper sign in equation (3.89)]

$$\sqrt{\alpha}[1 + O(\sqrt{\alpha} \ln \alpha)] = \sqrt{\frac{2\pi}{1 - \lambda/3}} \frac{3(1 + s_0)}{32} \left[\ln \frac{-(1 + s_0)}{64} + 3 \right], \qquad \alpha << 1 \quad (3.90)$$

It should be noted that for equation (3.89) a critical value $\alpha = \alpha_c$ exists which separates the α regions in which the upper and lower signs in equation (3.89) apply. For $\alpha < \alpha_c$ only in the case of the upper sign does a solution s_0 exist,[28] while for $\alpha > \alpha_c$ only the lower sign allows a solution s_0. This critical value α_c corresponds to the singularity $s_0(\alpha \to \alpha_c) \to -\infty$. Consequently, we find from (3.89) the equation for determining α_c by considering $s_0 \to -\infty$. It reads

$$1 + \left[1 + \frac{1}{w(\alpha_c)} - \frac{\alpha_c}{2\sqrt{2}} G(\alpha_c) \right] \left[1 + \frac{3(1 + \lambda)}{G(\alpha_c)} \right] = 0 \qquad (3.91)$$

Numerically, one finds $\alpha_c \simeq 0.70$ (see Fig. 3). Furthermore, there exists a maximal value $\alpha = \alpha_{\max} > \alpha_c$ beyond which no solution s_0 can be found. The value of $\alpha_{\max}$ corresponds to the limit $s_0(\alpha \to \alpha_{\max}) \to -1$. The corresponding equation for $\alpha_{\max}$ reads

$$\left[1 + \frac{4}{3\pi} \alpha_{\max} G(\alpha_{\max}) \right] + \left[1 + \frac{1}{w(\alpha_{\max})} - \frac{4}{3\pi} \alpha_{\max} G(\alpha_{\max}) \right] \left[1 + \frac{3(1 + \lambda)}{G(\alpha_{\max})} \right] = 0 \quad (3.92)$$

The numerical calculation yields $\alpha_{\max} \simeq 2.64$ (see Fig. 3).

From the above considerations it is clear that to find a consistent IR solution of the integral equation (3.42) requires the parameter β of our Ansatz (3.21) to be understood as some function of α and therefore it cannot be left arbitrary up to the point where we are going to impose the fixed-point condition for the kernel of the gauge field action. It will be true in general that a single parameter of any Ansatz (containing, say, n parameters) for the kernel of the gauge field action needs to be reserved to allow a consistent IR solution of the integral equation (3.42) to be found. We have only one

[28] It is clear that for small α (i.e., $\alpha \to 0$), a smooth transition from $\sqrt{-s_0}\, \tilde{a}_g(s_0) = \tilde{b}_g(s_0)$ to $\sqrt{-s_1}\, \tilde{a}(s_1) = \pm \tilde{b}(s_1)$ must exist; consequently the upper sign holds.

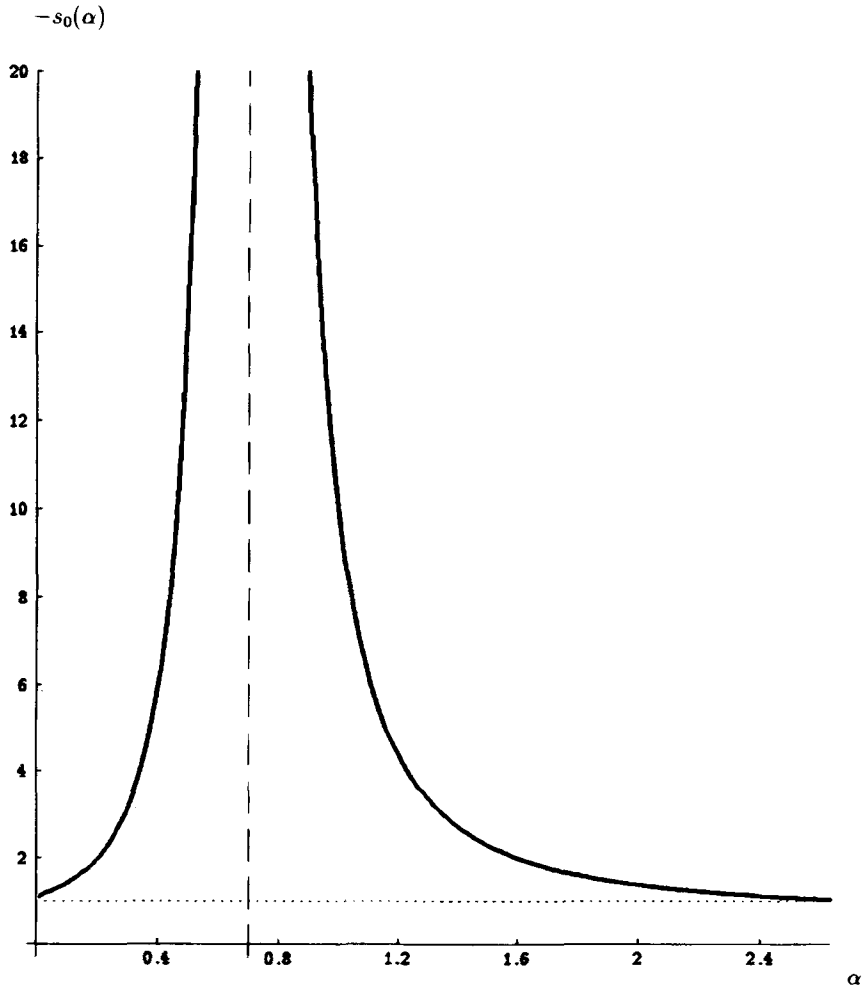

Fig. 3. Solution s_0 of equation (3.89) as a function of α. The dashed line is located at $\alpha_c \simeq 0.70$, corresponding to the singularity $-s_0 \to \infty$. It separates the regions where the upper ($\alpha < \alpha_c$) and the lower ($\alpha > \alpha_c$) sign in the normalization condition (3.85) are applied respectively. Beyond $\alpha_{max} \simeq 2.64$, equation (3.89) does not have any solution as one recognizes from the dotted line drawn at $-s_0 = 1$.

parameter at hand and from equation (3.76) we immediately find its dependence on α (see Fig. 4),

$$\beta = \beta(\alpha) = -\frac{\alpha^2 w(\alpha)^2}{4 s_0(\alpha)} \tag{3.93}$$

Here, $w(\alpha)$ and $s_0(\alpha)$ are solutions of equations (3.75) and (3.89), respectively. One easily recognizes (Fig. 4) that for small α the parameter β assumes

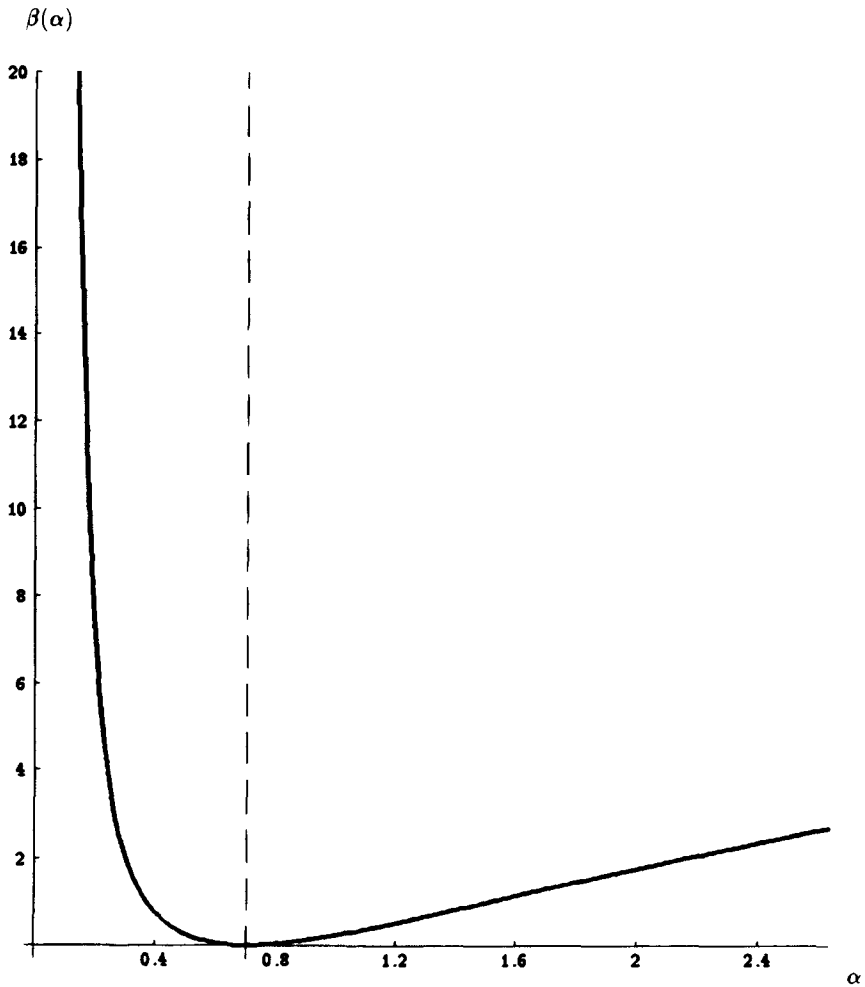

Fig. 4. The parameter β as a function of α [cf. equation (3.93)]. For further comments see Fig. 3.

unrealistically large values, which underscores the point that the present approximate calculation has to be understood as a model calculation only.

Having applied the normalization condition (3.85) and fixed the parameters G, s_0, β, we can write the functions $\tilde{a}$, $\tilde{a}_s$, $\tilde{b} = \tilde{b}(\infty) + \tilde{b}_s$ in the low-s region as follows ($s_0 \leq -1$):

$$\tilde{a}(s) = -\frac{T(-s/s_0, 0)}{sT(s_0^{-1}, 0)} \tag{3.94}$$

$$\tilde{b}(s) = [\pm 1 - \tilde{b}(\infty)]\tilde{a}(s) + \tilde{b}(\infty) \tag{3.95}$$

The parameter $\tilde{b}(\infty)$ in the normalization applied reads (see Fig. 5)

$$\tilde{b}(\infty) = \pm 1 - \sqrt{-s_0}\frac{H}{G} = \pm 1 + \sqrt{-s_0}\left[1 + \frac{3(1+\lambda)}{G(\alpha)}\right] \tag{3.96}$$

For small α we immediately find from equation (3.83)

$$\tilde{b}(\infty) = 1 + \sqrt{-s_0}\left[1 + 4(1+\lambda)\frac{\alpha}{\pi} + O(\alpha^{3/2})\right], \qquad \alpha << 1, \quad \alpha < \alpha_c \tag{3.97}$$

Taking into account equation (3.90) ($s_0 \simeq -1$, $\alpha << 1$), we recognize that for small α ($\alpha << 1$, $\alpha < \alpha_c$) we have $\tilde{b}(\infty) \simeq 2$. From a physical point of view this might be interpreted in such a way that at low energies the fermion action merely describes individual real fermions ($\tilde{b} \simeq 1$; i.e., a single-particle interpretation is possible), while at high energies it reflects collective properties of the vacuum, which are related to fermion (electron–positron) pairs; consequently, $\tilde{b} \sim \tilde{b}(\infty) \simeq 2$. Apparently, such an interpretation breaks down at stronger coupling.

Now, the appropriately normalized $\tilde{a}_g(s)$ [equation (3.74)] in the low-s region reads

$$\begin{aligned}\tilde{a}_g(s) = {} & \frac{C_g}{2\sqrt{\pi}}\Gamma\left(-\frac{1}{2} + \alpha\frac{3-\lambda}{2\pi}\right)(-s_0)^{-1/4} \\ & \times (\sqrt{-s_0} + \alpha/2\sqrt{\beta})^{1/2-\alpha(3-\lambda)/2\pi} \\ & \times \frac{T(s(\sqrt{-s_0} + \alpha/2\sqrt{\beta})^{-2}, \alpha(3-\lambda)/2\pi)}{sT(s_0^{-1}, 0)}\end{aligned} \tag{3.98}$$

and equation (3.78) can be written as

$$\tilde{b}_g(s) = [\pm 1 - \tilde{b}(\infty)]\left(1 + \frac{1}{w(\alpha)}\right)\tilde{a}_g(s) + \tilde{b}(\infty) \tag{3.99}$$

Clearly, s_0, β, and $\tilde{b}(\infty)$ are functions of α ($\lambda = 0$ as explained in Section 3.1).

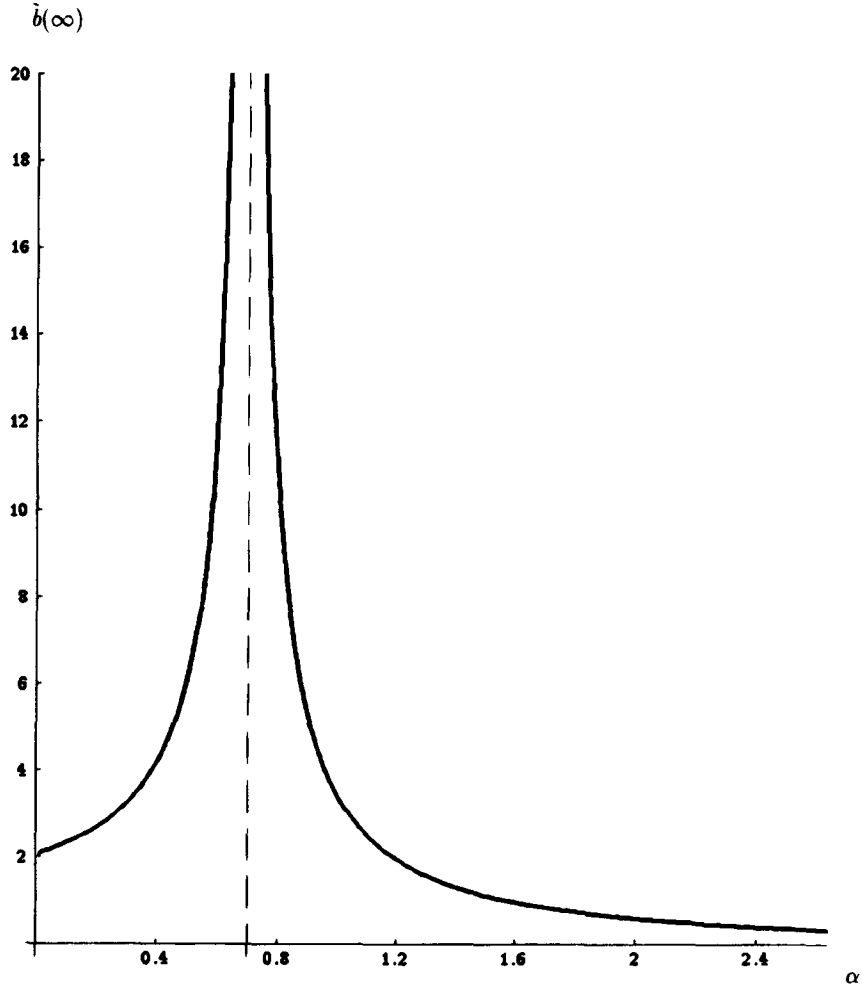

Fig. 5. The parameter $\tilde{b}(\infty)$ as a function of α [cf. equation (3.96)]. Note that $\tilde{b}(\infty)$ is close to 2 for small α. For further comments see Fig. 3.

Finally, the correctly normalized IR tails of a, b characterizing the kernel of the fermion action are

$$a(x_E) \overset{m^2 x_E^2 \to \infty}{=} \frac{m^4}{\sqrt{2}(2\pi)^{5/2}} \frac{(-s_0)^{-1/4}}{T(s_0^{-1}, 0)} \times (m|x_E|)^{-7/2}[\exp(-\sqrt{-s_0}\, m|x_E|)][1 + \cdots] \quad (3.100)$$

$$b(x_E) \overset{m^2x_E^2\to\infty}{=} [\pm 1 - \tilde{b}(\infty)]a(m^2x_E^2 \to \infty) \tag{3.101}$$

where s_0 and $\tilde{b}(\infty)$ are to be considered as functions of α. It is worth emphasizing that qualitatively equations (3.100) and (3.101) agree with the long-distance representation of the one-loop fermion self-energy calculated in standard QED perturbation theory.

To conclude this subsection, it should be emphasized that in analyzing the integral equation (3.42) for the kernel of the fermion action in the asymptotic UV and IR regions, respectively, based on certain reasonable assumptions, we have obtained a qualitative and nonperturbative understanding of the behavior of its solution. Furthermore, the IR analysis even yields approximate quantitative, nonperturbative results which, combined with the information about the UV behavior of the kernel of the fermion action, allows the approximate calculation of the QED coupling constant α to be attempted. This we will study now.

3.3.3. *The Fixed-Point Condition for the Kernel of the Gauge Field Action and the Approximate Calculation of the QED Coupling Constant* α

From equation (3.39) we recognize that the functional integration induces a change $\Delta\Gamma_I^G[A]$ to be added to the gauge field action $\Gamma_I^G[A]$ to obtain $\Gamma_{II}^G[A]$. In accordance with our approximation strategy, we display only those terms that match our Ansatz (3.21),

$$\Delta\Gamma_I^G[A] = \frac{\alpha}{4\pi}\int d^4x\, A^\mu(x)[g_{\mu\nu}\Box - \partial_\mu\partial_\nu]\left[C_{1a} + C_{2a}\frac{\Box}{m^2} + \cdots\right]A^\nu(x) \tag{3.102}$$

Because a_g, b_g respect conditions (3.15), (3.16) [cf. equations (3.58), (3.59)], no terms violating gauge invariance occur. C_{1a} reads explicitly [Scharnhorst (1995), equation (4.2)]

$$C_{1a} = \frac{2}{3}\ln\left[\frac{\tilde{b}(\infty)}{\tilde{b}_g(0)}\right]^2 - \int_0^\infty ds\, M(s) \tag{3.103}$$

$$\begin{aligned} M(s) = \frac{1}{s\tilde{a}_g^2 + \tilde{b}_g^2}\Bigg[&\frac{s\tilde{a}_g^2}{s\tilde{a}_g^2 + \tilde{b}_g^2}[s\tilde{a}_g\tilde{a}_g' + \tilde{b}_g\tilde{b}_g'] \\ &+ \frac{2}{3}s^3\tilde{a}_g\tilde{a}_g''' + 3s^2\tilde{a}_g\tilde{a}_g'' + \frac{2}{3}s^2\tilde{b}_g\tilde{b}_g''' \\ &+ 2s\tilde{a}_g\tilde{a}_g' + 3s\tilde{b}_g\tilde{b}_g'' - s(\tilde{b}_g')^2 + 3\tilde{b}_g\tilde{b}_g'\Bigg] \end{aligned} \tag{3.104}$$

(as explained in Section 3.3.1, we confine ourselves to one-loop contributions). From the above expression one recognizes that C_{1a} is an RG-invariant quantity, i.e., it is invariant against (finite) mass and (fermion) wave function renormalizations. C_{2a} has not yet been calculated in terms of $\tilde{a}_g$, $\tilde{b}_g$, but it will have an analogous representation. Because $\tilde{a}_g$, $\tilde{b}_g$ depend exclusively on α, the coefficients C_{1a}, C_{2a} can both be understood as functions of this parameter. Then the fixed-point condition $d_{\mathrm{I}} = d_{\mathrm{II}}$ according to our approximation strategy reads (cf. Section 3.1)

$$C_{1a}(\alpha) = 0 \tag{3.105}$$

$$C_{2a}(\alpha) = 0 \tag{3.106}$$

It is clear that within our approximative approach we do not have enough parameters left to satisfy both of these equations (unless they are degenerate, perhaps by accident). We decide to choose equation (3.105) as the fixed-point equation because we require that, at least in the asymptotic IR region, the fixed-point condition for the map f should be fulfilled. Consequently, to determine the QED coupling constant α we have to find the zero(s) of $C_{1a}(\alpha)$.

The explicit calculation of C_{1a} has, of course, to be based on information obtained in the preceding subsections. The first point to be made is that we will take equation (3.103) as it stands. In principle one could identically reformulate it by exploiting partial integrations for functions that obey conditions (3.15), (3.16). We choose the present representation for its 'minimal' shape (of course, this is merely a matter of taste). Let us also emphasize that it turns out to be advantageous because a certain piece is already integrated out and it therefore depends on the boundary values of $\tilde{b}_g$ only. This term contains certain nonperturbative information from the solution of the integral equation (3.42) for the kernel of the fermion action not easily incorporated otherwise. Finally, one should keep in mind that although different representations of equation (3.103) are equivalent in a rigorous mathematical sense, they may lead to different answers if approximate information is taken into account only (and this is what we will do).

Now, the first guess might be simply to insert into equation (3.103) the IR representation found for $\tilde{a}_g$, $\tilde{b}_g$ [equations (3.98), (3.99)]. However, it comes as no surprise that the integral in equation (3.103) is not convergent for $\alpha \leq \pi/3$ (it is logarithmically UV divergent then). In other words, this approximation would be so crude as to not even deliver finite results. So in the parameter region $\alpha \leq \pi/3$, at least, one has to proceed differently. Without any problem we may always insert the value of $\tilde{b}(\infty)$ determined by the normalization conditions applied within the IR analysis. For $\tilde{b}_g(0)$ and in the low-s integration region of the integral we will insert $\tilde{a}_g$, $\tilde{b}_g$ as given by equations (3.98), (3.99). In the large-s region $\tilde{a}_g$, $\tilde{b}_g$ will be taken from

equations (3.58), (3.59). One immediately recognizes that this is a better approximation because the integral in equation (3.103) then gives finite results. Now, of course, the practical question arises as to which intermediate value of the integration variable s in equation (3.103) should split the application regions of the IR and UV representations of $\tilde{a}_g$, $\tilde{b}_g$. Perhaps one could choose to fit together the IR and UV representations at some value of s to be determined by a certain condition. For the purpose of the present numerical calculation we select another way. The UV tail of the integrand $M(s)$ in equation (3.103) will not contribute significantly and we therefore ignore it by simply cutting the integration over the IR representation of the integrand at some upper value $s = s_x$. This value is determined as follows. Observe that the exact integrand $M(s)$ in equation (3.103) is positive for $s \to \infty$. To see this, one may insert equations (3.58), (3.59) into (3.104) and one finds to leading order

$$M(s) = 11 \frac{C_{\tilde{a}}^2}{s^4} + \cdots > 0, \qquad s \to \infty \tag{3.107}$$

On the other hand, one may easily convince oneself that for $\alpha \leq \pi/3$ the integrand $M(s)$ of equation (3.103) becomes negative for $s \to \infty$ if the low-s representations (3.98), (3.99) are inserted. One now detects that the integrand with the low-s representation inserted is positive for $s = 0$. Consequently, there exists a zero of the integrand taken in the IR representation (cf. Fig. 6). Obviously, this zero determines the point beyond which the IR (low-s) representation starts to strongly misrepresent the true integrand and we therefore choose this zero as the upper cutoff s_x of the numerical integration (see Fig. 7 for the dependence of s_x on α).[29] It is clear that this recipe leads to a certain slightly lower value of the integral than if the UV region was not neglected.

The result of the numerical calculation of $C_{1a}(\alpha)$ is shown in Fig. 8, while Fig. 9 displays the behavior of the two contributions from which $C_{1a}(\alpha)$ derives [see equation (3.103)]. Unfortunately, within the approximation applied we do not find any zero of $C_{1a}(\alpha)$, but from Fig. 9 one recognizes that both contributions to be taken into account are indeed comparable numerically. We believe that the contribution of the integral in equation (3.103) is underestimated within the approximation applied compared with the exact one, which relates to the exact solution of the integral equation (3.42). The contribution of the first term in equation (3.103) is probably determined to

[29] Another choice might be to fit the IR and UV representations of the integrand together at some $s_y < s_x$. Here, one way is to require continuity of the integrand at $s = s_y$ and to determine s_y by extremizing the value of the integral. However, in doing so, one detects that the contribution of the UV tail is negligible numerically.

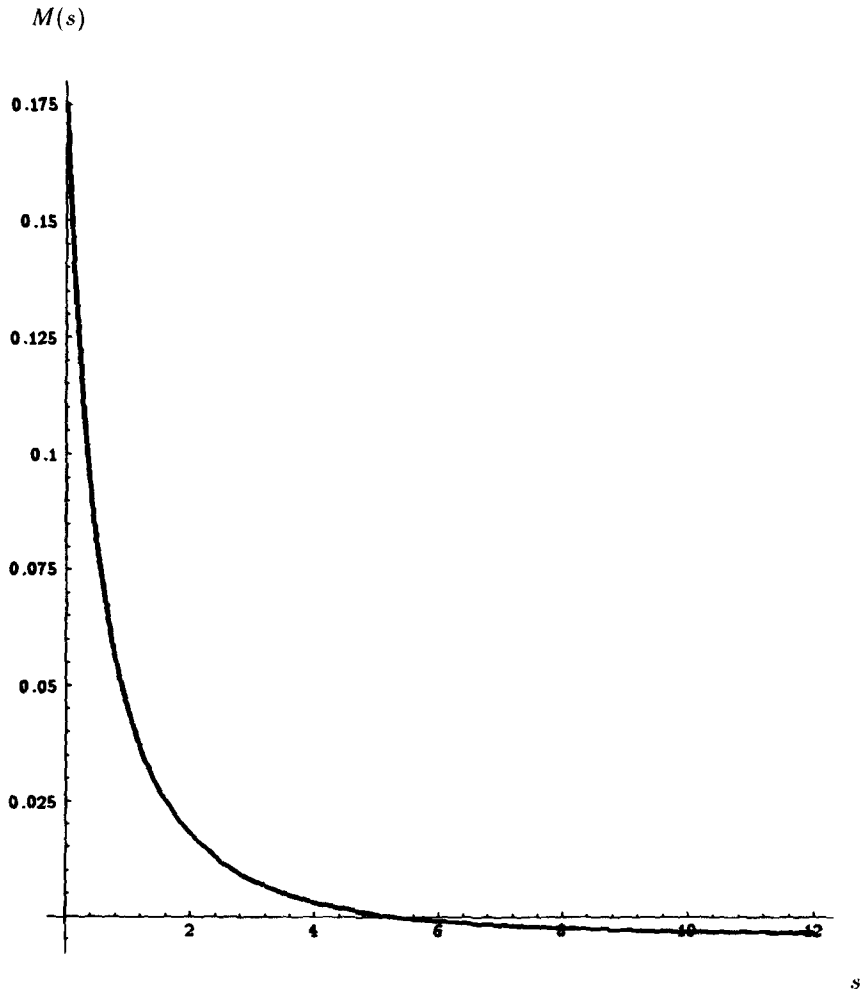

Fig. 6. Typical behavior of the integrand $M(s)$ in equation (3.103) for small arguments, where equations (3.98), (3.99) are inserted [$M(s)$ is drawn here for $\alpha = 0.05$]. The zero (we denote it by s_x) of the function $M(s)$ is understood as defining the applicability region of the low-s representation (3.98), (3.99). s_x as a function of α is shown in Fig. 7.

a more reliable degree because only the boundary values of $\tilde{b}_g(s)$ contribute to it. Furthermore, the smaller is α, the more the approximation applied for the second term in equation (3.103) miscalculates it. This can easily be seen from Fig. 6 (and Fig. 7). The true integrand [with the exact solution of equation (3.42) inserted, which we do not know presently] would likely contribute more because we expect the integrand $M(s)$ to be positive for large

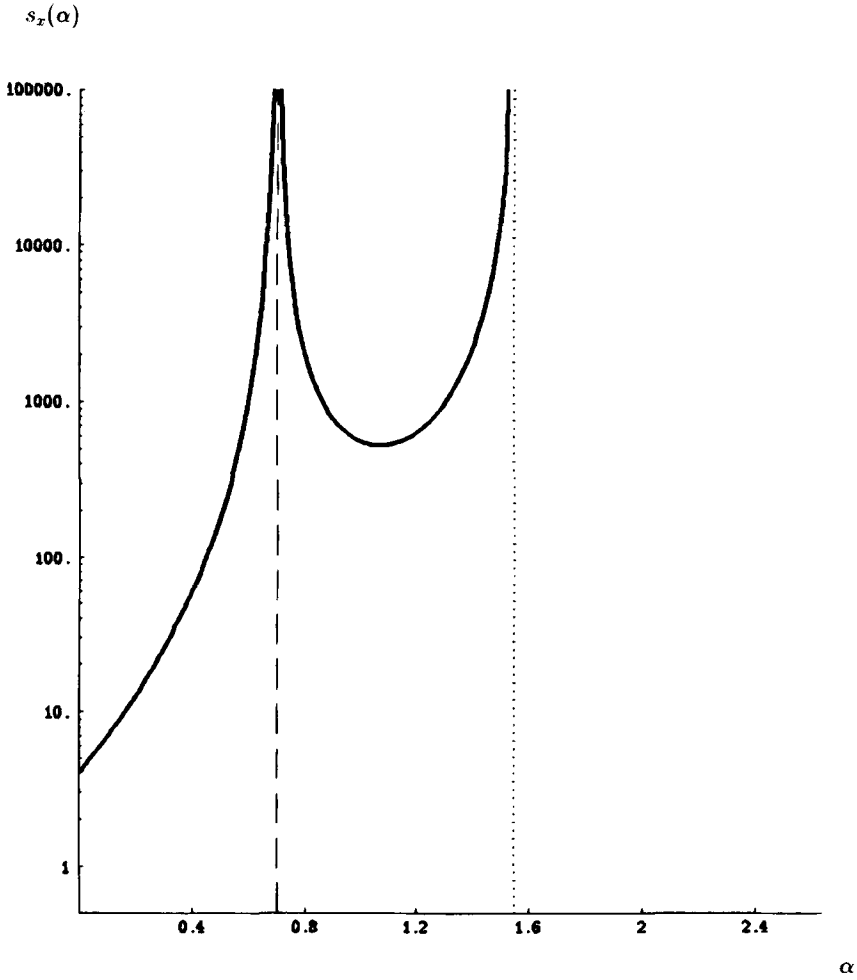

Fig. 7. Zero s_x of $M(s)$ as a function of α (see Fig. 6). The dashed line is drawn at α_c while the dotted line is located at $\alpha \simeq 1.55$. In both cases one finds numerically $s_x \to \infty$.

s. This would shift curve 2 in Fig. 9 to larger values and consequently a zero of $C_{1a}(\alpha)$ might occur.

In conclusion, the mechanism proposed has explicitly been shown capable to allow the calculation of the QED coupling constant α to be attempted. However, the approximation applied turns out to be too simple to yet obtain any specific value of α. In particular, for small values of α, where most of the approximations applied within the calculation given in the present section appear to be most justified, no zero of $C_{1a}(\alpha)$ is found. However, it is clear

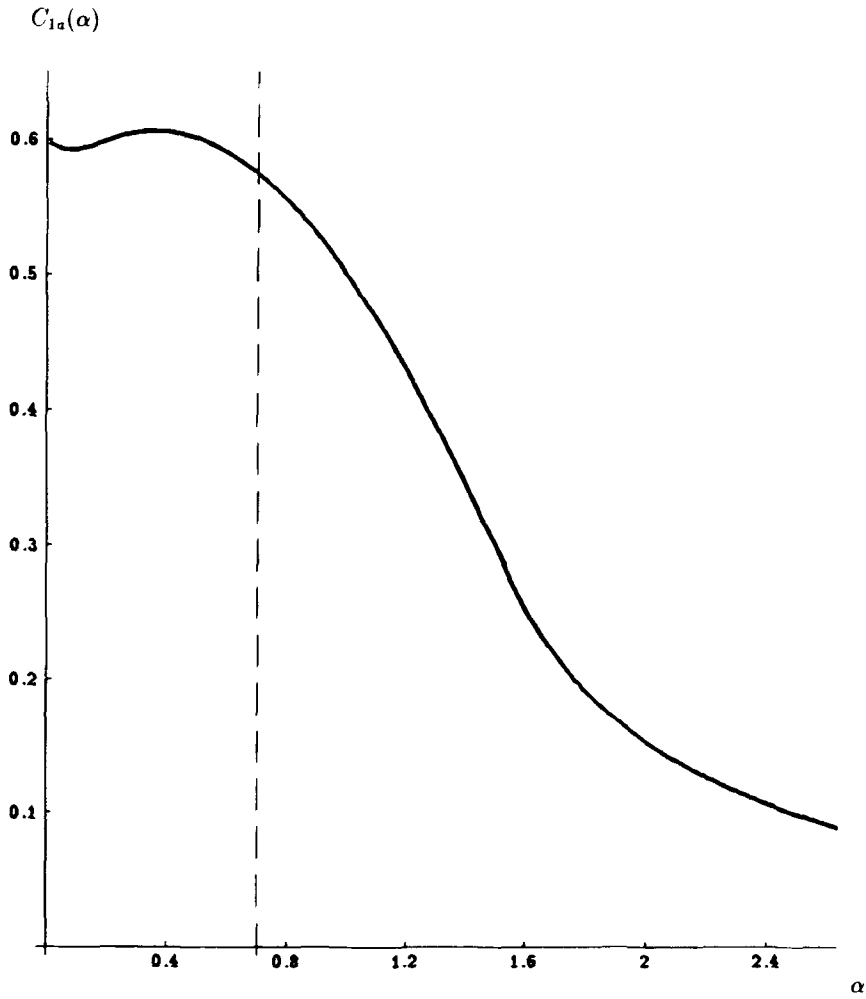

Fig. 8. The coefficient C_{1a} [see equation (3.103)] as a function of α. The curve is the difference of the contributions represented by curves 1 and 2 in Fig. 9 [curve 1 stands for the first term in equation (3.103), while curve 2 is the contribution of the integral]. Note that C_{1a} is a completely smooth function at $\alpha = \alpha_c$ (dashed line), although certain of the parameters (see Figs. 3 and 5) are singular there.

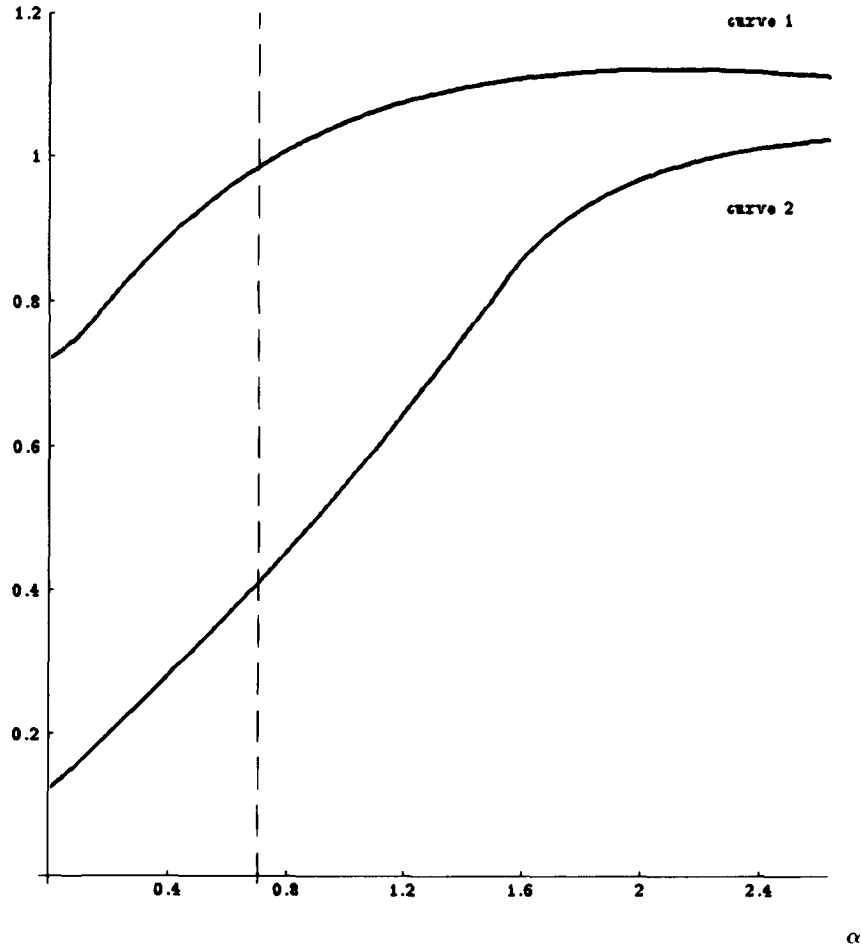

Fig. 9. Contributions to C_{1a} as functions of α. For a further explanation see Fig. 8.

that more advanced approximations may lead to a different picture. This needs to be studied in the future. We postpone further discussion of this issue to the next section.

4. DISCUSSION AND CONCLUSIONS

Before turning to some matters of principle, let us further discuss the approximative approach to the functional integral equation for the complete effective action of QED. We have seen that the general approximative approach chosen (Section 3.2) allows certain nonperturbative information

about the quadratic kernels of the QED action to be obtained. The fact that the information found indicates that there exists only a unique solution to the functional integral equation (at least within the approximative approach studied) deserves particular emphasis. Of course, this point has to be studied further using more advanced approximations in order to see whether for the QED coupling constant α only one admissible value exists (if any at all—but nature appears to allow for some). Furthermore, within the approximative approach divergences as they are characteristic for standard QED do not show up (at least, as far as the present study runs). It should perhaps also be said that the nonlocal character of the fermion action allows nonperturbative techniques to be employed which are not quickly applicable in standard QED. For example, as we have seen, the well-known Bloch–Nordsieck contribution can be obtained easily and it contains important IR (long-distance) information crucial to the further calculation. The present approach comes closest to standard QED for $\alpha_c \simeq 0.70$ because at this point the fermion form factors a, b become delta functions for all practical purposes, the parameter β vanishes, and only a finite photon wave function renormalization applies. In a certain sense, viewed from standard QED, it can be regarded for general α as a nonlocal regularization which is lifted when $\alpha \to 0.70$ (in standard quantum field theory language α_c corresponds to a fixed point of the renormalization group).

However, so far the concept proposed in the present article has not yet successfully passed the crucial test attempted in Section 3.3.3, namely the approximate calculation of the QED coupling constant α. As we have seen, the approach used is indeed suited for explicit calculation, but inasmuch as within the simple approximation applied we did not find any zero of $C_{1a}(\alpha)$, the question remains open. What might a better approximation look like? First, it should be noted that by imposing equation (3.75) independently of the value of α, a strong coupling condition has been enforced which annihilates the hope that higher loop contributions can really be neglected in the integral equation for the quadratic kernel of the fermion action (3.42). However, to take into account higher loop contributions would add complications to the formalism not easily resolved in analytical calculations. One way out of this dilemma might be to relax for approximation purposes the fixed-point condition for the quadratic kernel of the fermion action to $a_{II} = Ca_I$, $b_{II} = Cb_I$, where C is some arbitrary real constant, instead of immediately enforcing $C = 1$. This requirement of structural similarity perhaps could be sufficient to keep the conceptual content alive and at the same time count indeed in any argument on the eventual smallness of α [the point is that then one would not need to enforce equation (3.75)]. The parameter β then would also be unconstrained as long as the fixed-point condition $d_{II} = d_I$ is not enforced. To finally fix both α and the parameter β the conditions (3.23), (3.24)

can be applied simultaneously. Whether this recipe yields a more effective approximation remains to be seen in future investigations. It might perhaps also be necessary to include some higher loop contributions to C_{1a} and C_{2a}. Certainly, the solution of the integral equation for the kernel of the fermion action (3.42) has to be studied further. Maybe it will also be advisable to improve the Ansatz (3.21). These are some of the changes in the approximation strategy which can be implemented most easily along the lines of Section 3. Perhaps still more severe changes are required. Finally, it should be said that the calculation discussed in Section 3 should merely be understood as a first (naive) attempt to extract information out of the functional integral equation for the complete effective action by means of a simple approximation, which, however, allows mostly analytical investigation. It is clear, of course, that the present understanding is poor and much remains to be learned.

Let us turn now to the consideration of the vacuum energy problem within the present approach. It will not be aimed at the most general theoretical setup eventually possible, which very likely would turn out to be fruitless, but we restrict consideration to QED and in particular to the approximative approach to it studied in Section 3. It might be hoped that this special case yields certain new insight into the problem, useful at least for gauge field theories in general. In standard QED in 4D Minkowski space the vacuum energy density originating from fermion as well as from photon fluctuations and their interactions is a divergent quantity, but it is considered unimportant because it can either be removed by applying normal ordering (in operator quantization) or by appropriately normalizing the functional integral defining the theory. No physical quantity depends on it. However, it is also known that modifications of the vacuum energy density as occurring when external conditions are applied (boundary conditions, temperature, external fields) do matter and in certain cases consequences are even observable in experiment (e.g., the Casimir effect) (Plunien *et al.*, 1986; Grib *et al.*, 1988; Kapusta, 1989; Mostepanenko and Trunov, 1990). Some changes of the vacuum energy density turn out to be finite immediately (e.g., the Casimir energy density, or the free energy density for QED at finite temperature). Others require renormalization, like the QED effective potential for (say) a constant magnetic field. Even more care is needed in the study of QED in a gravitational background field, which we will turn to in a moment. However, a large part of the motivation for studying the vacuum energy density derives from this situation because it gives rise to the concept of induced (classical) gravity (Sakharov, 1967) understood as some kind of gravitational (metric) Casimir effect [for a review and further references see Adler (1982) and Novozhilov and Vassilevich (1991); also note David (1984) and David and Strominger (1984)].

First, let us compare the calculation of the vacuum energy density in standard QED and within the present approach whereby we restrict consideration to the one-loop level. The vacuum energy density ρ_{vac} is given within cutoff regularization by [cf. Appendix B, equation (B.7)]

$$\begin{aligned}&\Gamma_{II}[0, 0, 0]\\ &= -V_4\rho_{vac}\\ &= \text{const} + \frac{V_4}{8\pi^2}\, m^4 \int_0^{\Lambda^2/m^2} ds\, s\left\{\ln[s\tilde{a}_I(s)^2 + \tilde{b}_I(s)^2] - \frac{1}{2}\ln[s\tilde{d}_I(s)^{3/2}]\right\}\end{aligned} \tag{4.1}$$

One immediately recognizes the well-known fact that in standard QED ($\tilde{a}_I = \tilde{b}_I = \tilde{d}_I \equiv 1$) the vacuum energy density ρ_{vac} diverges. Now, QED in a background field (electromagnetic or gravitational; we restrict consideration to these external conditions that are most interesting in view of standard QED difficulties) will change the quantity s (stemming from differential operators in configuration space) appearing in the argument of the logarithms above to some $s + \Delta s$, where for large s the change Δs behaves like $\Delta s \overset{s\to\infty}{\simeq}$ const.[30] Of course, as already mentioned, one can always absorb the divergent terms characteristic for 4D Minkowski space displayed on the r.h.s. of equation (4.1) into the normalization constant of the functional integral. However, for QED in a background field the logarithm in the integrand of equation (4.1) then reads for large s

$$\ln\left[1 + \frac{\Delta s}{s} + \cdots\right] \overset{s\to\infty}{=} \ln[1 + O(s^{-1})] = O(s^{-1}) \tag{4.2}$$

and the vacuum energy density depending on the background field is still divergent (this even holds up to $\Delta s \overset{s\to\infty}{\simeq} 1/s$).

Now, compare this with our approximative approach to the equation for the complete effective action of QED. From equations (3.58), (3.59) we know that

$$s\tilde{a}_I(s)^2 + \tilde{b}_I(s)^2 \overset{s\to\infty}{=} \tilde{b}(\infty)^2\left[1 - \frac{C_{\tilde{a}}^2}{s^3} + \cdots\right] \tag{4.3}$$

Absorbing a $\ln \tilde{b}(\infty)$ term into the normalization constant of the functional integral, we see that the part of the vacuum energy density originating from fermion fluctuations [the first term in the integrand of equation (4.1)] is even finite without any further appeal to this constant. As we have explained in

[30] Considering a connection in the covariant derivatives, this naively yields $\Delta s \overset{s\to\infty}{\simeq} \sqrt{s}$, but symmetry reasons finally lead to the somewhat weaker behavior $\Delta s \overset{s\to\infty}{\simeq}$ const.

Section 3.3.2, this is true irrespective of the particular approximation applied (i.e., whether we first perform the gauge field integration or the fermionic integration). Consequently, any change of the fermionic part of the vacuum energy density under the influence of external (electromagnetic as well as gravitational) fields will also be finite. However, in view of condition (3.20), the part of the vacuum energy density originating from photon fluctuations [the second term in the integrand of equation (4.1)] is still divergent and the behavior of the gauge field determinant in the presence of a gravitational background field is of equal concern as in standard QED [in particular, the gauge field conformal anomaly and its regularization dependence (Birrell and Davies, 1982)]. It remains to be seen whether any still further modified approach can be found which improves the situation in the gauge field sector to the same extent as this seems to be the case in the fermion sector.

The above consideration now allows comparison of standard QED in a gravitational background field and the present approach. In standard QED the structure of the first few terms of the effective gravitational action [i.e., up to a minus sign, the (time-integrated) vacuum energy] is known (Birrell and Davies, 1982; Grib *et al.*, 1988; Fulling, 1989),

$$\Gamma_{II}[0, 0, 0] = \int d^4x \sqrt{-g}\{m^4 c_1 + m^2 c_2 R + c_3 \Box R + c_4 R^2 + c_5 R_{\mu\nu} R^{\mu\nu} + c_6 R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} + \cdots\} \tag{4.4}$$

c_1 to c_6 are certain divergent dimensionless constants. We have already discussed c_1 (i.e., $-\rho_{vac}$ for 4D Minkowski space); c_2 is a quadratically (in the cutoff Λ) divergent quantity; while c_3 to c_6 diverge logarithmically. All further terms are finite. Consistency requires that we start in the standard QED functional integral with a certain bare gravitational action (included in Γ_I) containing all terms displayed in equation (4.4) in order to be able to absorb the divergences into the bare constants in front of them. Consequently, induced gravity is not a consistent concept within standard QED. In contrast to standard QED, by taking into account the UV behavior of the quadratic kernel of the fermion action (a consequence of the equation for the complete effective action of QED), we have demonstrated above that whatever the technical approach to calculate c_2 to c_6 will be in detail,[31] these coefficients will come out finite (at least at the one-loop level). However, the contribution from the determinant of the gauge field kernel remains a problem. Irrespective of this, viewed from the fermionic sector, within the present framework induced gravity might have a better chance of turning out to be a valid

[31] In cutoff regularization they will have representations analogous to equations (2.7)–(2.9) in Scharnhorst (1995).

concept. Of course, as pointed out by Sakharov (1967) in his pioneering paper, the (induced) gravitational action will very likely not be dominated by contributions stemming from QED, but from the heaviest excitations (particles) existent in nature. If one would like to attempt the calculation of the induced gravitational action within the concept proposed in the present paper, one would first have to study the equation for the complete effective action of the standard model at least. If one is willing to do so, this will require much effort and certainly results cannot be obtained quickly. However, in view of the possible outcome perhaps it might be worthwhile.

Throughout the paper we have preliminarily applied the standard point of view that the space-time structure is prescribed to the functional integral equation for the complete effective action. In a certain sense it is considered as 'classical' and as prior to quantum effects (at least for flat space-time). However, the criticism spelled out in Section 2 with respect to the artificial distinction between classical action and effective action also applies to this view on the space-time structure. Therefore, the structure of space-time should be more adequately understood as some characteristics of the quantum field-theoretic vacuum. Basically, this is the point of view applied within the concept of induced gravity, although this aspect is hardly discussed in the literature. However, in flat space-time the idea also applies. Recent investigations of the propagation of light in a Casimir vacuum indicate that this concept is already implicit in standard QED (Scharnhorst, 1990; Barton, 1990; Barton and Scharnhorst, 1993). As discussed in Barton and Scharnhorst (1993), although the lack of appropriate nonperturbative calculational tools leaves the question so far unsettled in the strict sense, the only conceptually viable (as far as present knowledge is concerned) of the alternatives allowed by the Kramers–Kronig relation for the refractive index $n(\omega)$ of the Casimir vacuum (ω is the frequency of the test wave) is that $n(\infty) < 1$ holds for the propagation of light perpendicular to two parallel mirrors in the slab between them [this entails a signal (front) velocity of light larger than in the unbounded space vacuum]. While this result is often viewed as something like a paradox in standard QED, it is easily understandable by means of the concept put forward in the present article (where it may count as a special application). If the map f is modified in such a way that it is no longer fully Lorentz invariant,[32] then the solution of the functional integral equation for the complete effective action is also no longer fully Lorentz invariant and the dispersion analysis in accordance with the effective Maxwell action may well reveal a change in the signal velocity of light. The point is that only one situation can be considered as the one where normalization is performed (and we typically

[32]For an appropriate functional integral formulation of standard QED in the presence of two parallel mirrors see Bordag *et al.* (1985).

choose free Minkowski space as the reference situation and the signal velocity of light there as the reference standard, although of course any less symmetrical setup could also be used). However, in view of the discussion in Section 2, it makes no sense to consider any normalized value of a certain quantity (e.g., mass, charge, velocity of light) as classical, because this is a concept not accessible to experiment. We can only denote certain values defined by a certain measurement scenario under defined circumstances as reference values. Any changes of these values measured under different circumstances are certainly of a quantum nature, but equally well these values could have served as initial reference values. Consequently, it appears most sensible to consider these quantities from the very beginning as characteristics of the quantum field-theoretic vacuum and their changes as parametrizing changes of it with respect to some reference situation.

Summarizing the concept proposed in the present article, let us point out that it proposes a view of quantum field theory which differs from the established one, but the established standard paradigm finds it natural explanation and place within this new approach. In particular, it incorporates and continues in modified shape certain ideas used in local renormalizable quantum field theory, such as the unobservability of bare quantities and the hypothesis that the vanishing of the beta function(s) (corresponding to a fixed point of the renormalization group) defines the physical coupling constant(s) of a model. The functional integral equation for the complete effective action proposed ensures (merely by definition) that any of its solutions is finite (otherwise it is not a solution). This removes to a certain extent the concern of divergences that standard quantum field theory is beset by, but the price to pay for this is the present uncertainty of whether the functional integral equation proposed has any other nontrivial solution beyond free field theories [i.e., any nonlinear (interacting) field theory]. The most natural place to find out whether the proposed concept is physically correct should be QED, because unlike some other model theories, it is a theory of phenomena definitely present in nature. QED is certainly structurally more complex than, say, scalar model field theories, but if for QED something new can be learned, we may feel sure that our physical understanding has advanced. The approximative approach to the functional integral equation for the QED effective action presented has proved its calculational accessibility. Although the particular approximation studied is still quite simple, it has yielded certain nonperturbative information which indicates that the present approach also has certain calculational advantages. However, only further investigation will show whether any obviously appropriate approximation can be found which yields the correct value of the fine structure constant with reasonable calculational effort. In a certain sense this should be viewed as a crucial test because

in principle the present approach should be able to pass it, if it is really physically correct and adequate.

APPENDIX A

In this appendix we explicitly calculate the function

$$g(x - x') = \exp\left[-\frac{i}{2} \int d^4y\, d^4y'\, \bar{J}_\mu(x, x'; y) D_{\rm I}^{\mu\nu}(y, y') \bar{J}_\nu(x, x'; y') \right] \tag{A.1}$$

for the Ansatz (3.21),

$$d_{\rm I}(x) = \left[1 + \beta \frac{\Box}{m^2} \right] \delta^{(4)}(x) \tag{A.2}$$

Equation (A.1) can easily be rewritten as[33]

$$g(x - x') = \exp\left[-ie^2 (x - x')^2 \int_0^1 d\tau\, (1 - \tau) D_{\rm I}((x - x')\tau) \right]$$
$$\times \exp\{ ie^2 (1 - \lambda)[D_{\rm I}^*(x - x') - D_{\rm I}^*(0)] \} \tag{A.3}$$

where[34]

$$D_{\rm I}^*(x) = \int \frac{d^4p}{(2\pi)^4} \frac{e^{ipx}}{(p^2 + i\epsilon)^2} \frac{1}{\tilde{d}_{\rm I}(p)}, \qquad \tilde{d}_{\rm I}(p) = 1 - \beta \frac{p^2}{m^2} \tag{A.4}$$

and

$$D_{\rm I}(x) = \Box D_{\rm I}^*(x) \tag{A.5}$$

For simplicity, let us perform the calculation for g in Euclidean space. Results then can be read off for Minkowski space whenever needed by rotating back the fourth coordinate. In Euclidean space $D_{\rm I}^*$ and $D_{\rm I}$ read

$$D_{\rm I}^*(x_{\rm E}) = -\frac{i}{16\pi^2} \ln(\mu^2 x_{\rm E}^2) - \frac{\beta}{m^2} D_{\rm I}(x_{\rm E}) \tag{A.6}$$

(μ^2 is the temporary IR cutoff applied) and

[33] Of course, this transformation is not specific to the Ansatz (A.2). To obtain equation (A.3), a gauge-fixing term Γ_{gf} with $\bar{n}_\mu = ip_\mu\, \tilde{d}_{\rm I}(p)^{1/2}$ has been added to the gauge field action $\Gamma_{\rm I}^G$.

[34] The IR divergence can be regularized and then drops out for $g(x)$. The spurious pole generated by the model Ansatz $\tilde{d}_{\rm I}(p)$ is understood as also supplied with the $i\epsilon$ prescription.

$$D_1(x_E) = \frac{i}{4\pi^2 x_E^2} - \frac{im}{4\pi^2\sqrt{\beta}\,|x_E|} K_1\left(\frac{m|x_E|}{\sqrt{\beta}}\right) \tag{A.7}$$

For the further calculation the following integral turns out to be useful ($\mathbf{L}_\nu$ are Struve functions) (Prudnikov *et al.*, 1990–1992, Vol. 2):

$$\int \frac{d\tau}{\tau} K_1(\tau) = -K_1(\tau) - \tau K_0(\tau) - \frac{\pi}{2}\tau[K_1(\tau)\mathbf{L}_0(\tau) + K_0(\tau)\mathbf{L}_1(\tau)] \tag{A.8}$$

Consequently, we find (γ is the Euler constant)

$$\begin{aligned} -x_E^2 \int_0^1 d\tau\,(1-\tau) D_1(x_E\tau) &= \frac{im}{4\pi^2}\left\{1 + \gamma + \frac{1}{2}\ln\left(\frac{m^2x_E^2}{4\beta}\right)\right. \\ &+ \left(1 - \frac{m^2x_E^2}{\beta}\right)K_0\left(\frac{m|x_E|}{\sqrt{\beta}}\right) - \frac{m|x_E|}{\sqrt{\beta}} K_1\left(\frac{m|x_E|}{\sqrt{\beta}}\right) - \frac{\pi}{2}\frac{m^2x_E^2}{\beta} \\ &\left.\times\left[K_1\left(\frac{m|x_E|}{\sqrt{\beta}}\right)\mathbf{L}_0\left(\frac{m|x_E|}{\sqrt{\beta}}\right) + K_0\left(\frac{m|x_E|}{\sqrt{\beta}}\right)\mathbf{L}_1\left(\frac{m|x_E|}{\sqrt{\beta}}\right)\right]\right\} \end{aligned} \tag{A.9}$$

The final result for $g(x_E)$ is then ($t = m|x_E|/\sqrt{\beta}$)

$$\begin{aligned} g(x_E) = \exp\left(\frac{\alpha}{\pi}\right.&\left\{1 + \gamma + \frac{1}{2}\ln\frac{t^2}{4} + (1-t^2)K_0(t) - tK_1(t)\right. \\ &\left.\left. - \frac{\pi}{2}t^2[K_1(t)\mathbf{L}_0(t) + K_0(t)\mathbf{L}_1(t)]\right\}\right. \\ &\left. + \frac{\alpha}{\pi}(1-\lambda)\left[\frac{1}{t^2} - \frac{1}{t}K_1(t) + \frac{1}{4}(2\gamma - 1) + \frac{1}{4}\ln\frac{t^2}{4}\right]\right) \end{aligned} \tag{A.10}$$

In the long-distance limit ($t >> 1$) equation (A.10) reads

$$g(x_E) = \exp\left\{\frac{\alpha}{2\pi}\left[-\pi t + \frac{3-\lambda}{2}\ln\frac{t^2}{4} + \frac{3+\lambda}{2} + (3-\lambda)\gamma + \cdots\right]\right\} \tag{A.11}$$

So in the long-distance region in which we are mainly interested the function $g(x_E)$ can be written as

$$g(x_E) = C_g\,(m^2 x_E^2)^{\alpha(3-\lambda)/4\pi}\left[\exp\left(-\frac{\alpha}{2\sqrt{\beta}}\,m|x_E|\right)\right][1 + \cdots] \quad \text{(A.12)}$$

$$C_g = (4\beta)^{-\alpha(3-\lambda)/4\pi}\exp\left\{\frac{\alpha}{4\pi}[(3+\lambda) + 2(3-\lambda)\gamma]\right\} \quad \text{(A.13)}$$

One easily recognizes in equation (A.12) the well-known exponent of the (power-like) Bloch–Nordsieck contribution (Kernemann and Stefanis, 1989; Karanikas *et al.*, 1992; and references therein).

APPENDIX B

In this appendix we derive an expression for the vacuum energy density on the one-loop level for the version of nonlocal QED given by equations (3.12)–(3.14). We apply the simplest regularization possible, namely cutoff regularization [with a (radial) momentum space UV cutoff at Λ], which is most suited for our purposes. The vacuum energy density ρ_{vac} is given by

$$\Gamma_{II}[0,0,0] = -V_4\rho_{vac}$$

$$= \text{const} - i\ln \text{Det}_\Lambda(S_I^{-1}) - i\ln \text{Det}_\Lambda(D_{gh,I}^{-1}) + \frac{i}{2}\ln \text{Det}_\Lambda(D_{i,\mu\nu}^{-1}) \quad \text{(B.1)}$$

Here

$$S_I^{-1}(x-x') = i\,\partial\!\!\!/_x a_I(x-x') - m b_I(x-x') \quad \text{(B.2)}$$

$$D_{gh,I}^{-1}(x-x') = \frac{1}{\sqrt{\lambda}}\,{}_x\partial_\mu n^\mu(x-x') \quad \text{(B.3)}$$

$$D_{I,\mu\nu}^{-1}(x-x') = [g_{\mu\nu}\,{}_x\Box - {}_x\partial_\mu\,{}_x\partial_\nu]\,d_I(x-x') - \frac{1}{\lambda}\int d^4y\, n_\mu(y-x)n_\nu(y-x') \quad \text{(B.4)}$$

are the quadratic kernels of the fermion, ghost (contributing in QED to the vacuum energy only), and gauge field actions, respectively. n_μ here can be any vector-valued distribution, for example, perhaps a derivative ∂_μ acting on some scalar function leading to a Lorentz-type gauge, or any constant vector times a scalar function yielding an axial type gauge. From equation (B.1) we find

$$\Gamma_{II}[0,0,0] = \text{const} - 2iV_4\int_\Lambda \frac{d^4p}{(2\pi)^4}\ln[-p^2\tilde{a}_I(p)^2 + m^2\tilde{b}_I(p)^2]$$

$$- iV_4\int_\Lambda \frac{d^4p}{(2\pi)^4}\ln[i\lambda^{-1/2}p\tilde{n}(p)]$$

$$+\frac{i}{2}V_4\int_\Lambda \frac{d^4p}{(2\pi)^4}\ln\{\det[(g_{\mu\nu}p^2-p_\mu p_\nu)\,\tilde{d}_1(p)-\lambda^{-1}\tilde{n}_\mu(p)\tilde{n}_\nu(p)]\} \tag{B.5}$$

Taking into account the relation

$$\det[(g_{\mu\nu}p^2-p_\mu p_\nu)\tilde{d}-\lambda^{-1}\tilde{n}_\mu\tilde{n}_\nu]=-\frac{\tilde{d}^3}{\lambda}[p\tilde{n}]^2[p^2]^2 \tag{B.6}$$

and applying a Wick rotation, one finds after some manipulations [we have absorbed certain ln m terms into the first (normalization) constant on the r.h.s. of equation (B.5)]

$$\Gamma_{II}[0,0,0]=\text{const}+\frac{V_4}{8\pi^2}m^4\int_0^{\Lambda^2/m^2} ds\,s\left\{\ln[s\tilde{a}_1(s)^2+\tilde{b}_1(s)^2]-\frac{1}{2}\ln[s\tilde{d}_1(s)^{3/2}]\right\} \tag{B.7}$$

There is no trace left of the gauge condition because we have correctly included in the kernel of the ghost action (B.3) the gauge parameter λ [for a related discussion see Allen and Ottewill (1992) and Nielsen and van Nieuwenhuizen (1988)].

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